1. Consider the linear system $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2-\varepsilon^2 \\ 2-2\varepsilon \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1-\varepsilon \end{bmatrix}$$

The inverse A^{-1} of A is given by

$$A^{-1} = \varepsilon^{-2} \begin{bmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{bmatrix}$$

Compute for $\varepsilon = 10^{-1}, ..., 10^{-6}$ the corresponding solution x of the above system by

(a) using the Gaussian elimination

(b) direct multiplication $A^{-1}b$.

2. Consider the *sparse* matrix $A = (a_{ij})_{1\leq i,j\leq 10}$ with $a_{ij} \neq 0$ for $i = 1$, $j = 1, ..., 10$, as well as $j = 1$, $i = 1, ..., 10$ and $i = j$, $j =, ..., 10$; otherwise $a_{ij} = 0$.

(a) Sketch the *sparsity pattern* of the such a matrix, i.e. where are the non-zero entries.

(b) Verify that the first step of the Gaussian elimination elimination without pivoting transform zeroentries into non-zero entries.

(c) Show that we can avoid transforming zero entries into non-zero entries, by simply interchanging rows and/or columns of the matrix A before applying the Gaussian elimination.

Find the solution of the following linear systems usin

(i) Naive Gaussian elimination.

(ii) Gaussian elimination with scaled partial pivoting.

3. $$\begin{array}{rrrrl} 2x_1 & -3x_2 & 4x_3 & x_4 & = 10 \\ & 7x_2 & 4x_3 & 3x_4 & = 6 \\ & & 5x_3 & 0x_4 & = -5 \\ & & & -4x_4 & = -4 \end{array}$$

4. $$\begin{array}{rrrl} x_1 & -2x_2 & x_3 & = -1 \\ -3x_1 & -x_2 & 4x_3 & = -2 \\ 2x_1 & 4x_2 & 0x_3 & = -3 \end{array}$$

5. $$\begin{array}{rrrrl} x_1 & -4x_2 & x_3 & x_4 & = 8 \\ -3x_1 & 8x_2 & 2x_3 & 2x_4 & = -4 \\ -2x_1 & -2x_2 & 5x_3 & x_4 & = 2 \\ -4x_1 & -3x_2 & 2x_3 & -5x_4 & = -1 \end{array}$$

6. $$\begin{array}{rrrrl} 2x_1 & -3x_2 & 4x_3 & x_4 & = 12 \\ x_1 & 7x_2 & 4x_3 & 3x_4 & = -6 \\ -2x_1 & -2x_2 & 5x_3 & 0x_4 & = 4 \\ 2x_1 & -3x_2 & 4x_3 & -4x_4 & = -3 \end{array}$$

7. Solve the system

$$\begin{array}{rrrrl} 4x_1 & 2x_2 & x_3 & 0x_4 & = 11 \\ -2x_1 & 4x_2 & -x_3 & 7x_4 & = -4 \\ -5x_1 & 2x_2 & 3x_3 & x_4 & = -3 \\ 6x_1 & 4x_2 & -3x_3 & 2x_4 & = 2 \end{array}$$

using Gaussian elimination with scaled partial pivoting. Show intermediate matrices at each step.

8. Consider the system of equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & k \\ 2k & 1 \end{bmatrix}, \quad k \text{ real.}$$

   (a) find the value of k for which the matrix A is strictly diagonally dominant.

   (b) For $k = 0.25$, solve the system using The Jacobi method.

9. Consider the following system of equations

$$\begin{array}{rrrl} 4x_1 & 0x_2 & 2x_3 & = 4 \\ 0x_1 & 5x_2 & 2x_3 & = -3 \\ 5x_1 & 4x_2 & 10x_3 & = 2 \end{array} .$$

   Set up the Jacobi and Gauss-Seidel iterative schemes for the solution and iterate three times starting with the initial vector $\mathbf{x}^{(0)} = \mathbf{0}$.

   Find the solution of the following linear systems $Ax = b$ using the LU decomposition method.

10. $$\begin{bmatrix} -3 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -2.4 \end{bmatrix}$$

11. $$\begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12.3 \\ 14.2 \end{bmatrix}$$

12. $$\begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & -2 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10.0 \\ -3.4 \\ 2.7 \end{bmatrix}$$

13. $$\begin{bmatrix} 6 & -5 & 0 & 1 \\ 3 & 2 & 1 & -4 \\ -5 & -6 & 0 & 2 \\ -1 & 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11.9 \\ -8.4 \\ 14.2 \\ 6.3 \end{bmatrix}$$

14. Given the symmetrix matrix

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

   (a) Show that A is positive definite.

   (b) Determine an lower-triangular matrix L such that $LL^T = A$

Computer Assigents

1. In the structure shown, six wires support three beams. Three weights are attached at the points shown. Assume that th structure is stationary and that the weights of the wires and beams are very small compared to the applied weights. The principles of statics state that the sum of the forces is zero and that the sum of the moments about any point is zero. Applying these principles to each beam using the free-body diagrams shown, we obtain the following equations.

   Beam 1

   $T_1 + T_2 = T_3 + T_4 + W_1 + T_6$

   $-T_3 - 4T_4 - 5W_1 - 6T_6 + 7T_2 = 0$

   Beam 2

   $T_3 + T_4 = W_2 + T_5$

   $-W_2 - 2T_5 + 3T_4 = 0$

   Beam 3

   $T_5 + T_6 = W_3$

   $-W_3 + 3T_6 = 0$

   Use the MATLAB function `ngaussel.m` to determine the forces T_i for the following cases.

   (a) $W_1 = W_2 = W_3 = 0\ N$

   (b) $W_1 = W_2 = W_3 = 300\ N$

   (c) $W_1 = 400\ N, W_2 = 300\ N, W_3 = 200\ N$

   Summarize these results in a table.

2. Create a 10×10 matrix A with random interger entries $a_{ij} \in [-6, 6]$, $1 \leq i, j \leq 10$. To this end, make use of the MATLAB function `randint`. Use the MATLAB function `lufact` to find the solution of the linear system of equations $A\mathbf{x} = \mathbf{b}$ for the following coefficient vectors:

(a) $\mathbf{b} = [1, 1, ..., 1]'$

(b) $\mathbf{b} = [1, 2, 3, 4, 5, 0.1, 0.2, 0.3, 0.4, 0.5]'$

(c) $\mathbf{b} = [2, 4, 6, 8, 10, 1, 3, 5, 7, 9]'$

3. Consider solving $A\mathbf{x} = \mathbf{b}$ for x, where A an $n \times n$ nonsingular matrix. There are two obvious algorithms. The first algorithm factorizes $PA = LU$ using Gaussian elimination and then solves for each column of X by forward and backward substitution. The secaond algorithm computes A^{-1} using Gaussian elimination and then multiplies $X = A^{-1}\mathbf{b}$. Use the MATLAB commands '\' (backslash) and inv. Record the CPU seconds required by each algorithm. Observe that the first one is faster.

4. Use the MATLAB function gaussel.m to find the solution of the following linear system:

$$\begin{bmatrix} 0.378 & 0.6321 & 0.0662 & 0.2457 & 0.1281 & 0.4414 \\ 0.6924 & 0.7461 & 0.2286 & 0.8044 & 0.3281 & 0.5646 \\ 0.7920 & 0.2994 & 0.5044 & 0.5787 & 0.0826 & 0.2399 \\ 0.7758 & 0.3962 & 0.4622 & 0.4058 & 0.1713 & 0.3482 \\ 0.9570 & 0.6141 & 0.5165 & 0.4767 & 0.3346 & 0.8807 \\ 0.0432 & 0.8976 & 0.5470 & 0.2036 & 0.8721 & 0.2812 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 12.3428 \\ -10.6701 \\ 6.9823 \\ 8.4511 \\ 3.0012 \\ -13.1223 \end{bmatrix}$$

5. A classical example of an ill-conditioned matrix is the so-called Hilbert matrix. This is a symmetric matrix of the form

$$H = h_{ij} = \begin{bmatrix} 1 & 1/2 & ... & 1/n \\ 1/2 & 1/3 & ... & 1/(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ 1/n & 1/(n+1) & ... & 1/(2n) \end{bmatrix}$$

Define $b_i = \sum_{j=1}^{n} h_{ij}$ so that the solution of the linear sysem $Hx = b$ is $\mathbf{x} = [1, 1, ..., 1]^T$. Use the MATLAB function gaussel.m to solve the system for $n = 5, 10$ and 15 and compare the solutions vector with the exact ones. Verify that the solution vector gets overhelmed by roundoff error as n increases.

6. Use the MATLAB function seidel.m with $\mathbf{x}^{(0)} = \mathbf{0}$ to solve the linear system

$$\begin{bmatrix} 2 & -5 & 2 & 5 & 30 & -8 \\ -8 & 36 & -1 & 0 & 1 & 8 \\ 11 & -4 & 25 & 1 & -4 & 4 \\ 1 & 0 & -3 & 19 & 2 & 1 \\ 42 & -2 & -9 & 0 & 3 & 0 \\ -3 & 7 & -7 & 4 & 5 & -32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 27 \\ 36 \\ 42 \\ 18 \\ 54 \\ 60 \end{bmatrix}.$$

At first glance the set does not seem to be suitable for an iterative solution, since the coefficient matrix is not diagonally dominant. However, by simply reordering the equations the matrix can be made diagonally dominant. Use the MATLAB function seidel.m with $\mathbf{x}^{(0)} = \mathbf{0}$ to solve the new reodered system and compare the number of iterations needed for convergence for each case.

7. Repeat the preceding computer problem using the MATLAB function jacobi.m.

8. The linear system

$$\begin{bmatrix} 10^{-6} & 10^{-6} & 1 \\ 10^{-6} & -10^{-6} & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10^{-6} \\ -2 \cdot 10^{-6} \\ 1 \end{bmatrix}$$

has the exact solution

$$x_1 = \frac{-1}{1 - 2 \cdot 10^{-6}}, \quad x_2 = 2, \quad x_3 = \frac{10^{-6}}{1 - 2 \cdot 10^{-6}}$$

Solve the system using four-digits floating point arithmetic

(a) without pivoting.

(b) with complete pivoting.

9. The following system has an approximate solution $x_1 = 3.072$, $x_2 = -5.497$, $x_3 = -2.211$ and $x_4 = 4.579$.

$$\begin{matrix} 1.20 & 0.45 & 0.35 & 0.45 \\ 0.89 & 2.59 & -0.33 & -0.22 \\ 0.71 & 0.78 & 4.01 & -0.88 \\ 0.11 & 0.55 & 0.66 & 3.39 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = \begin{matrix} 2.500 \\ -11.781 \\ -15.002 \\ 11.378 \end{matrix} .$$

Use both the Gauss-seidel and Jacobi methods to the approximate solution of the system.