9.3 THEOREM. Chain Rule. *If* $\underline{r}$ *is a differentiable vector function and* s = u(t) *is a differentiable scalar function then the derivative of* $\underline{r}$(s) *with respect to t is*

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds} . \frac{ds}{dt}$$

9.4 THEOREM. Rules of Differentiation. *Let* $\underline{r}_1$ *and* $\underline{r}_2$ *be differentiable vector functions and* u(t) *a differentiable scalar function. Then*

(i) $\frac{d}{dt}[\underline{r}_1(t) + \underline{r}_2(t)] = \underline{r}_1'(t) + \underline{r}_2'(t)$;

(ii) $\frac{d}{dt}[u(t)\underline{r}_1(t)] = u(t)\underline{r}_1'(t) + u'(t)\underline{r}_2(t)$;

(iii) $\frac{d}{dt}[\underline{r}_1(t) \bullet \underline{r}_2(t)] = \underline{r}_1(t) \bullet \underline{r}_2'(t) + \underline{r}_1'(t) \bullet \underline{r}_2(t)$;

(iv) $\frac{d}{dt}[\underline{r}_1(t) \times \underline{r}_2(t)] = \underline{r}_1(t) \times \underline{r}_2'(t) + \underline{r}_1'(t) \times \underline{r}_2(t)$.

Integrals of Vector Functions. *If f, g and h are integrable then the indefinite and definite integrals of a vector function* $\underline{\mathrm{r}}(\mathrm{t}) = f(t)\underline{\mathrm{i}} + g(t)\underline{\mathrm{j}} + h(t)\underline{\mathrm{k}}$ *are defined, respectively, by*

$$\int \underline{\mathrm{r}}(t)\,dt = \left[\int f(t)dt\right]\underline{i} + \left[\int g(t)dt\right]\underline{j} + \left[\int h(t)dt\right]\underline{k}$$
$$\int_a^b \underline{r}(t)\,dt = \left[\int_a^b f(t)dt\right]\underline{i} + \left[\int_a^b g(t)dt\right]\underline{j} + \left[\int_a^b h(t)dt\right]\underline{k}$$

The indefinite integral of $\underline{\mathrm{r}}(\mathrm{t})$ *is another vector function* $\underline{\mathrm{R}}(\mathrm{t}) + c$ *such that* $\underline{\mathrm{R}}(\mathrm{t}) = \underline{\mathrm{r}}(\mathrm{t})$.

Length of a Space Curve. *If function* $\underline{\mathrm{r}}(\mathrm{t}) = f(t)\underline{\mathrm{i}} + g(t)\underline{\mathrm{j}} + h(t)\underline{\mathrm{k}}$ *is a smooth function then it can be shown that the* length *of the smooth curve traced by* $\underline{\mathrm{r}}(\mathrm{t})$ *is given by*

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \|\underline{r}'(t)\|dt.$$

Example 8*(p.456) Integral of a Vector Function

If $\underline{\mathrm{r}}(\mathrm{t}) = t\,\underline{\mathrm{i}} + 3t^2\,\underline{\mathrm{j}} + 4t^3\,\underline{\mathrm{k}}$ *then*

$$\int_{-1}^{2} \underline{r}(t)dt = \int_{-1}^{2} (t\,\underline{i} + 3t^2\,\underline{j} + 4t^3\,\underline{k})dt$$
$$= \left[\int_{-1}^{2} t dt\right]\underline{i} + \left[\int_{-1}^{2} 3t^2 dt\right]\underline{j} + \left[\int_{-1}^{2} 4t^3 dt\right]\underline{k}$$
$$= \left[t^2/2\right]_{-1}^{2} + \left[t^3\right]_{-1}^{2} + \left[t^4\right]_{-1}^{2}$$
$$= \frac{3}{2}\underline{i} + 9\underline{j} + 15\underline{k}.$$

Example 9*(pp.456-7) Length of a Space Curve

Find the length of the curve traced by $\underline{\mathrm{r}}(\mathrm{t}) = t\,\underline{\mathrm{i}} + t\cos t\,\underline{\mathrm{j}} + t\sin t\,\underline{\mathrm{k}}$ $(0 \le t \le p)$.

Solution.

$$s = \int_0^p \|\underline{r}'(t)\|\,dt = \int_0^p \|\underline{i} + (\cos t - t\sin t)\underline{j} + (\sin t - t\cos t)\underline{k}\|\,dt$$
$$= \int_0^p \sqrt{1 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\,dt = \int_0^p \sqrt{2 + t^2}\,dt$$
$$= \frac{p\sqrt{2 + p^2}}{2} + \sinh^{-1}(p\sqrt{2}/2).$$ *(Integration by parts then using integral table).*

9.5 Directional Derivative

The Gradient of a Function. *The vector differential operator called* del- *or* nabla- operator *is given by*

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} \quad or \quad \nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

When the del-operator is applied to a differentiable function $z = f(x, y)$ *or* $w = F(x, y, z)$*, we say that the vectors*

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} \quad and \quad \nabla F(x, y, z) = \frac{\partial F}{\partial x}\underline{i} + \frac{\partial F}{\partial y}\underline{j} + \frac{\partial F}{\partial z}\underline{k}$$

are the gradients *of f and F, respectively.* ∇f *is usually read:* grad *f.*

Example 1* (p.474) Gradient

9.5 DEFINITION Directional Derivative

The directional derivative *of* $z = f(x, y)$ *in the direction of a unit vector* $\underline{u}(\theta) = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}$ *is*

$$D_{\underline{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

provided the limit exists.

9.6 THEOREM. Computing a Directional Derivative. *If* $z = f(x, y)$ *is a differentiable function of* x *and* y *and* $\underline{u} = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}$, *then*

$$D_{\underline{u}} f(x,y) = \nabla f(x,y) \bullet \underline{u} \qquad (5)$$

Maximum Value of the Directional Derivative *Let f represent a function of either two or three variables. Since (5) and its three-variable analogue express the directional derivative as a dot product, we see from Definition 7.3 (MATH 201) that*

$$D_{\underline{u}} f = \| \nabla f \| \, \| \underline{u} \| \cos f = \| \nabla f \| \cos f, \quad (\| \underline{u} \| = 1),$$

where f *is the angle between* ∇f *and* $\underline{\mathrm{u}}$. *Because* $0 \le f \le p$ *we have* $-1 \le \cos f \le 1$ *and, consequently,* $-\| \nabla f \| \le D_{\underline{u}} f \le \| \nabla f \|$. *In other words:*

> *The maximum value of the directional derivative is* $\| \nabla f \|$ *and it occurs when* $\underline{\mathrm{u}}$ *has the same direction as* ∇f $(\textit{when} \cos f = 1)$,

and:

> *The minimum value of the directional derivative is* $-\| \nabla f \|$ *and it occurs when* $\underline{\mathrm{u}}$ *and* ∇f *have opposite directions* $(\textit{when} \cos f = -1)$.

9.7 Divergence and Curl

Vector Fields *Vector functions of two and three variables,*

$\underline{F}(x, y) = P(x, y)\,\underline{i} + Q(x, y)\,\underline{j}$
$\underline{F}(x, y, z)) = P(x, y, z)\,\underline{i} + Q(x, y, z)\,\underline{j} + R(x, y, z)\,\underline{k}$

are also called vector fields.

9.7 DEFINITION Curl

The Curl *of a vector field* $\underline{F} = P\,\underline{i} + Q\,\underline{j} + R\,\underline{k}$ *is the vector field*

$$curl\ \underline{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\underline{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\underline{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k}$$

9.8 DEFINITION Divergence

The Divergence *of a vector field* $\underline{F} = P\,\underline{i} + Q\,\underline{j} + R\,\underline{k}$ *is the scalar function*

$$div\ \underline{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$