King Fahd University of Petroleum and Minerals

Dhahran 31261

Department of Mathematical Sciences

MATH 301 –02 & 04 (062)

Major Exam II

Monday - April 2nd, 2007

Instructor: Dr. A. Umar

Duration	**1 hr 40 mins**
Time	20.10 – 21.50 hrs

Name: ______________________________________

Student ID:________________ **Encircle Your Section: 02 or 04** (minus 5 points if you don't)

Notes:
1. Students must have a valid KFUPM ID Card with them.
2. You must show all your work to justify your answers.
3. Be as organized as possible. Only neat and logical solutions will attract points.
4. Programmable calculators and mobile phones are NOT allowed.
5. Scratch papers are attached at the end of this question paper. **PLEASE, DO NOT REMOVE THEM.**

PAGE	Points
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4	
5	
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7	
Total	

Q1. Find the length of the curve traced by the vector function: [10 pts]

$$\underline{r}(t) = t\,\underline{i} + t\cos t\,\underline{j} + t\sin t\,\underline{k}$$

on the interval $[0, \pi]$.

$$= f(t)\,\underline{i} + t\cos t\,\underline{j} + t\sin t\,\underline{k}$$

Arc length is given by

$$s = \int_0^{\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_0^{\pi} \sqrt{1 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2 + t^2}\, dt$$

$$= \left[\frac{t}{2}\sqrt{2+t^2} + \sinh^{-1}\left(\frac{t}{\sqrt{2}}\right) \right]_0^{\pi}$$

$$= \frac{\pi}{2}\sqrt{2+\pi^2} + \sinh^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

Q2. Evaluate div(curl $\underline{F}$) when $\underline{F}(x, y, z) = 5y^3\,\underline{i} + (\frac{1}{2}x^3y^2 - xy)\,\underline{j} - x^3yz\,\underline{k}$. [10 pts]

$$\text{Curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5y^3 & \frac{1}{2}x^3y^2 - xy & -x^3yz \end{vmatrix} = (-x^3z)\,\underline{i} + (+3x^2yz)\,\underline{j} + (\tfrac{3}{2}x^2y^2 - y - 15y^2)\,\underline{k}$$

and

$$\text{div}(\text{Curl } \underline{F}) = \nabla\cdot(\text{Curl } \underline{F})$$

$$= -3x^2z + 3x^2z + 0$$

$$= 0.$$

Q3. Find the work done by a force $\underline{F}(x, y) = (2x + e^{-y})\,\underline{i} + (4y - xe^{-y})\,\underline{j}$ along the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ from (2, 0) to (–2, 0) in the positive direction. [14 pts]

$$W = \int_C \underline{F} \cdot d\underline{r}$$

$$= \int_C (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy$$

(let $x = 2\cos t$, $y = 3\sin t$ then $dx = -2\sin t\,dt$, $dy = 3\cos t\,dt$)

$x = 2, y = 0 \Rightarrow t = 0$ and $x = -2, y = 0 \Rightarrow t = \pi$

$$= \int_0^\pi (4\cos t + e^{-3\sin t})(-2\sin t\,dt) + (12\sin t - 2\cos t\,e^{-3\sin t})(3\cos t\,dt)$$

$$= \int_0^\pi (-4\sin 2t - 2\sin t\,e^{-3\sin t} + 18\sin 2t - 6\cos^2 t\,e^{-3\sin t})\,dt$$

$$= \int_0^\pi 14\sin 2t\,dt \quad 2\int_0^\pi (-\sin t)\,e^{-3\sin t}\,dt - 6\int_0^\pi \cos^2 t\,e^{-3\sin t}\,dt$$

$$= \left[-7\cos 2t\right]_0^\pi + 2\cos t\,e^{-3\sin t}\Big|_0^\pi + \cancel{6\int_0^\pi \cos^2 t\,e^{-3\sin t}\,dt} - \cancel{6\int_0^\pi \cos^2 t\,e^{-3\sin t}\,dt}$$

$$= [-7 - (-7)] + [-2 - 2]$$

$$= -4.$$

OR

$$W = \int_C (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy$$

Let $P = 2x + e^{-y}$ and $Q = 4y - xe^{-y}$. Then

$\frac{\partial P}{\partial y} = -e^{-y} = \frac{\partial Q}{\partial x}$ (i.e., the integral is path independent)

So there exist $\phi(x, y)$ such that $\frac{\partial \phi}{\partial x} = P$ and $\frac{\partial \phi}{\partial y} = Q$.

From $P = \frac{\partial \phi}{\partial x} \Rightarrow \phi = \int (2x + e^{-y})\,dx = x^2 + xe^{-y} + g(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = -xe^{-y} + g'(y) = Q = 4y - xe^{-y}$

$\Rightarrow g'(y) = 4y \Rightarrow g(y) = y^2$

Hence $\phi(x, y) = x^2 + xe^{-y} + y^2$. Therefore

$W = \phi(-2, 0) - \phi(2, 0) = (\cancel{4} - 2) - (\cancel{4} + 2) = -4.$

Q4. Use Green's Theorem to evaluate $\oint_C e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy$ where C is the ellipse: $9(x-1)^2 + 4(y-3)^2 = 36$. [8 pts]

Here $P(x,y) = e^{2x}\sin 2y$ and $Q(x,y) = e^{2x}\cos 2y$. Thus

$$\frac{\partial P}{\partial y} = 2e^{2x}\cos 2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2e^{2x}\cos 2y.$$

By Green's theorem

$$\oint_C e^{2x}\sin 2y\,dx + e^{x}\cos 2y\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R 0\,dA = 0.$$

Q5. In Example 1 on page 526 it is shown that the surface area A(S) of the portion of the sphere: $x^2 + y^2 + z^2 = a^2$ that is above the xy-plane and within the cylinder: $x^2 + y^2 = b^2, 0 < b \le a$ is $2\pi a(a - \sqrt{a^2 - b^2})$ square units. Use this to find the total surface area of a sphere of radius a. [8 pts]

If $b = a$ then the portion of the sphere that is above the xy-plane and within the cylinder is a hemisphere whose surface area is

$$A(S)\big|_{b=a} = 2\pi a^2$$

Hence the total surface area of the sphere is $2(2\pi a^2) = 4\pi a^2$ sq. units, where a is the radius of the sphere.

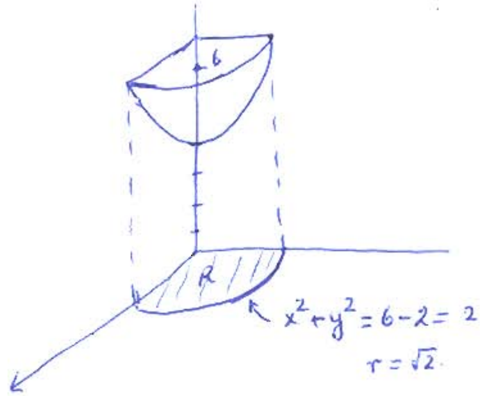
Q6. Find the mass of the surface of the paraboloid: $x^2 + y^2 = z - 4$ in the first octant for $4 \le z \le 6$ if the density at point P on the surface is directly proportion to the square of its distance from the xy-plane. Assume that $\int_0^a r(4+r^2)^2\sqrt{1+4r^2}\,dr = 2g(a)$

[14 pts]

The density $\rho = kz^2$ (for some constant k), and so the mass is

$$m = \iint_S \rho\, dS = \iint_S kz^2\, dS$$

$$= k\iint_R (4+x^2+y^2)^2\sqrt{1+4x^2+4y^2}\, dA$$



$$\begin{bmatrix} z = 4+x^2+y^2 \\ z_x = 2x \\ z_y = 2y \end{bmatrix}$$

(Changing to polar coordinates)

$$= k\int_0^{\pi/2}\int_0^2 (4+r^2)^2\sqrt{1+4r^2}\; r\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\left(\int_0^2 r(4+r^2)^2\sqrt{1+4r^2}\,dr\right)d\theta$$

$$= k\int_0^{\pi/2} 2g(\sqrt{2})\,d\theta$$

$$= k.2g(\sqrt{2}).[\theta]_0^{\pi/2}$$

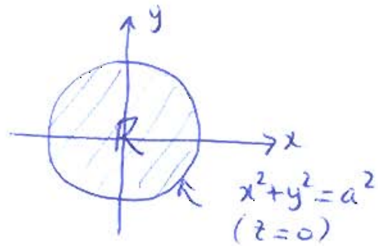
$$= k\pi g(\sqrt{2}).$$

Q5. Alternative Solution.

$$x^2+y^2+z^2=a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2} = f(x,y)$$

and so

$$f_x = \frac{-x}{\sqrt{a^2-x^2-y^2}} \quad \text{and} \quad f_y = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$



Now the total surface area A(S) of a sphere of radius a is

$$A(S) = 2\iint_S dS = 2\iint_R \sqrt{1+f_x^2+f_y^2}\, dA$$

$$= 2a\iint_R \frac{1}{\sqrt{a^2-x^2-y^2}}\, dA = 2a\int_0^{2\pi}\int_0^a (a^2-r^2)^{-1/2}\, r\,dr\,d\theta \quad \text{(In polar coordinates)}$$

$$= -2a\int_0^{2\pi} (a^2-r^2)^{1/2}\Big|_0^a\, d\theta$$

$$= 2a^2[\theta]_0^{2\pi} = 4\pi a^2.$$

Q7. Use Stokes' Theorem to evaluate $\oint_C \underline{F} \bullet d\underline{r}$ where [16 pts]

$$\underline{F}(x, y, z) = y^3\,\underline{i} - x^3\,\underline{j} + z^3\,\underline{k},$$

and C is the trace of the cylinder $x^2 + y^2 = 1$ in the plane $x + y + z = 1$. Assume C is oriented in the positive direction as viewed from above.

Let $g(x,y,z) = x + y + z - 1 = 0$ and $z = 1 - x - y = f(x,y)$. 2

Then

$$\underline{n} = \frac{\nabla g}{\|g\|} = \frac{\underline{i} + \underline{j} + \underline{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k}) \quad 2$$

and

$$\text{Curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -x^3 & z^3 \end{vmatrix} = 0\underline{i} + 0\underline{j} + (-3x^2 - 3y^2)\underline{k} \quad 2$$

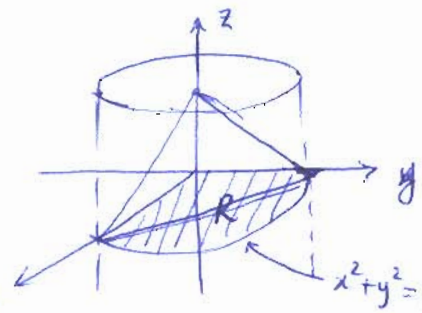
Thus

$$\text{Curl } \underline{F} \cdot \underline{n} = -\frac{3}{\sqrt{3}}(x^2 + y^2) = -\sqrt{3}(x^2 + y^2) \quad 2$$

By Stokes' theorem

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_S (\text{Curl } \underline{F} \cdot \underline{n})\, dS \quad 2$$

$[f_x = -1,\ f_y = -1]$



$$= -\sqrt{3} \iint_R (x^2 + y^2)\sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= -\sqrt{3} \iint_R (x^2 + y^2)\sqrt{3}\, dA \quad 2$$

(Changing to polar coordinates)

$$= -3 \int_0^{2\pi} \int_0^1 r^2 (r\,dr\,d\theta)$$

$$= -3 \int_0^{2\pi} \left(\int_0^1 r^3\,dr \right) d\theta$$

$$= -3 \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^1 d\theta$$

$$= -\frac{3}{4} \int_0^{2\pi} d\theta = -\frac{3}{4}(2\pi) = -\frac{3\pi}{2}$$

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