9.3 THEOREM. Chain Rule. *If* $\underline{r}$ *is a differentiable vector function and* s = u(t) *is a differentiable scalar function then the derivative of* $\underline{r}$(s) *with respect to* *t is*

$$\frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds} . \frac{ds}{dt}$$

9.4 THEOREM. Rules of Differentiation. *Let* $\underline{r}_1$ *and* $\underline{r}_2$ *be differentiable vector functions and* u(t) *a differentiable scalar function. Then*

(i) $$\frac{d}{dt}[\underline{r}_1(t) + \underline{r}_2(t)] = \underline{r}_1'(t) + \underline{r}_2'(t);$$

(ii) $$\frac{d}{dt}[u(t)\underline{r}_1(t)] = u(t)\underline{r}_1'(t) + u'(t)\underline{r}_2'(t);$$

(iii) $$\frac{d}{dt}[\underline{r}_1(t) \bullet \underline{r}_2(t)] = \underline{r}_1(t) \bullet \underline{r}_2'(t) + \underline{r}_1'(t) \bullet \underline{r}_2(t);$$

(iv) $$\frac{d}{dt}[\underline{r}_1(t) \times \underline{r}_2(t)] = \underline{r}_1(t) \times \underline{r}_2'(t) + \underline{r}_1'(t) \times \underline{r}_2(t).$$

Integrals of Vector Functions. *If f, g and h are integrable then the indefinite and definite integrals of a vector function* $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ *are defined, respectively, by*

$$\int \underline{r}(t)\,dt = \left[\int f(t)dt\right]\underline{i} + \left[\int g(t)dt\right]\underline{j} + \left[\int h(t)dt\right]\underline{k}$$
$$\int_a^b \underline{r}(t)\,dt = \left[\int_a^b f(t)dt\right]\underline{i} + \left[\int_a^b g(t)dt\right]\underline{j} + \left[\int_a^b h(t)dt\right]\underline{k}$$

The indefinite integral of $\underline{r}(t)$ *is another vector function* $\underline{R}(t) + c$ *such that* $\underline{R}(t) = \underline{r}(t)$.

Length of a Space Curve. *If function* $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ *is a smooth function then it can be shown that the* length *of the smooth curve traced by* $\underline{r}(t)$ *is given by*

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \|\underline{r}'(t)\|dt.$$

Example 8*(p.456) Integral of a Vector Function

If $\underline{r}(t) = t\,\underline{i} + 3t^2\,\underline{j} + 4t^3\,\underline{k}$ *then*

$$\int_{-1}^2 \underline{r}(t)dt = \int_{-1}^2 (t\,\underline{i} + 3t^2\,\underline{j} + 4t^3\,\underline{k})dt$$
$$= \left[\int_{-1}^2 t\,dt\right]\underline{i} + \left[\int_{-1}^2 3t^2 dt\right]\underline{j} + \left[\int_{-1}^2 4t^3 dt\right]\underline{k}$$
$$= [t^2/2]_{-1}^2 + [t^3]_{-1}^2 + [t^4]_{-1}^2$$
$$= \frac{3}{2}\underline{i} + 9\,\underline{j} + 15\underline{k}.$$

Example 9*(pp.456-7) Length of a Space Curve

Find the length of the curve traced by $\underline{r}(t) = t\,\underline{i} + t\cos t\,\underline{j} + t\sin t\,\underline{k}$ $(0 \le t \le p)$.

Solution.

$$s = \int_0^p \|\underline{r}'(t)\|\,dt = \int_0^p \|\underline{i} + (\cos t - t\sin t)\underline{j} + (\sin t - t\cos t)\underline{k}\|\,dt$$
$$= \int_0^p \sqrt{1 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\,dt = \int_0^p \sqrt{2 + t^2}\,dt$$
$$= \frac{p\sqrt{2 + p^2}}{2} + \sinh^{-1}(p\sqrt{2}/2).$$ *(Integration by parts then using integral table).*

9.5 Directional Derivative

The Gradient of a Function. *The vector differential operator called* del- *or* nabla- operator *is given by*

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} \quad or \quad \nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

When the del-operator is applied to a differentiable function $z = f(x, y)$ *or* $w = F(x, y, z)$, *we say that the vectors*

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} \quad and \quad \nabla F(x, y, z) = \frac{\partial F}{\partial x}\underline{i} + \frac{\partial F}{\partial y}\underline{j} + \frac{\partial F}{\partial z}\underline{k}$$

are the gradients *of f and F, respectively.* ∇f *is usually read:* grad *f.*

9.5 DEFINITION Directional Derivative

The directional derivative *of* $z = f(x, y)$ *in the direction of a unit vector* $\underline{u}(q) = \cos q\, \underline{i} + \sin q\, \underline{j}$ *is*

$$D_{\underline{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos q, y + h\sin q) - f(x, y)}{h}$$

provided the limit exists.

9.6 THEOREM. Computing a Directional Derivative. *If* $z = f(x, y)$ *is a differentiable function of* x *and* y *and* $\underline{u} = \cos q\, \underline{i} + \sin q\, \underline{j}$, *then*

$$D_{\underline{u}} f(x, y) = \nabla f(x, y) \bullet \underline{u} \qquad (5)$$

Maximum Value of the Directional Derivative *Let f represent a function of either two or three variables. Since (5) and its three-variable analogue express the directional derivative as a dot product, we see from Definition 73 (MATH 201) that*

$$D_{\underline{u}} f = \| \nabla f \| \, \| \underline{u} \| \cos f = \| \nabla f \| \cos f, \quad (\| \underline{u} \| = 1),$$

where f *is the angle between* ∇f *and* $\underline{\mathrm{u}}$. *Because* $0 \le f \le p$ *we have* $-1 \le \cos f \le 1$ *and, consequently,* $-\| \nabla f \| \le D_{\underline{u}} f \le \| \nabla f \|$. *In other words:*

> *The maximum value of the directional derivative is* $\| \nabla f \|$ *and it occurs when* $\underline{\mathrm{u}}$ *has the same direction as* ∇f $(\textit{when } \cos f = 1)$,

and:

> *The minimum value of the directional derivative is* $-\| \nabla f \|$ *and it occurs when* $\underline{\mathrm{u}}$ *and* ∇f *have opposite directions* $(\textit{when } \cos f = -1)$.

Example 4 (pp. 476-7) Directional Derivative

Example 8 (pp. 478) Direction to Cool Off Fastest

9.7 Divergence and Curl

Vector Fields *Vector functions of two and three variables,*

$$\underline{F}(x, y) = P(x, y)\,\underline{i} + Q(x, y)\,\underline{j}$$
$$\underline{F}(x, y, z)) = P(x, y, z)\,\underline{i} + Q(x, y, z)\,\underline{j} + R(x, y, z)\,\underline{k}$$

are also called vector fields.

9.7 DEFINITION Curl

The Curl *of a vector field* $\underline{F} = P\,\underline{i} + Q\,\underline{j} + R\,\underline{k}$ *is the vector field*

$$curl\,\underline{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\underline{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\underline{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k}$$

9.8 DEFINITION Divergence

The Divergence *of a vector field* $\underline{F} = P\,\underline{i} + Q\,\underline{j} + R\,\underline{k}$ *is the scalar function*

$$div\,\underline{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Reading assignment: 'Physical Interpretations'' on pages 486-7.

Definitions. (p. 487)

(i) *If div(*F*(P)) > 0, then P is said to be a* source *for* F*;*

(ii) *If div(*F*(P)) < 0, then P is said to be a* sink *for* F*;*

(iii) *If* $\nabla \bullet$ F *= 0, the fluid is said to be* incompressible*;*

(iv) *If* $\nabla \bullet$ F *= 0, the vector field* F *is said to be* solenoidal.

9.8 Line Integrals

The notion of the definite integral $\int_a^b f(x)dx$, *that is, integration of a function defined over an interval, can be generalized to integration of a function defined along a curve. To this end we need some terminology about curves.*

Definitions/Terminology. *Suppose C is a curve parametrized by x = f(t), y = g(t),* $a \le t \le b$, *and A and B are the points (f(a), g(a)) and (f(b), g(b), respectively we say that:*

- *(i)* *C is a* smooth curve *if f' and g' are continuous on the closed interval [a, b] and not simultaneously zero on the open interval (a, b);*
- *(ii)* *C is* piecewise smooth *if it consists of a finite number of smooth curves* $C_1, C_2, \ldots, C_n$ *joined end to end,---- that is,* $C = C_1 \cup C_2 \cup \ldots \cup C_n$;
- *(iii)* *C is a* closed curve *if A = B;*
- *(iv)* *C is a* simple closed curve *if A = B and it does not cross itself;*
- *(v)* *If C is not a closed curve, then the* positive direction *on C is the direction corresponding to increasing values of t.*

Line Integrals in the Plane (*z = G(x, y))*.

Example 1* (p. 491) Evaluation of Line Integral.

A line integral along a piecewise smooth curve C is defined as the sum of the integrals over the various smooth curves whose union comprises C. For example, if C is composed of smooth curves C_1 and C_2, then

$$\int_C G(x,y)ds = \int_{C_1} G(x,y)ds + \int_{C_2} G(x,y)ds$$

In many applications, line integrals appear as a sum

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy$$

A line integral along a closed curve C is very often denoted by

$$\oint_C P(x,y)dx + Q(x,y)dy$$

Example 4* (pp. 492-3) Closed Curve Defined by an Explicit Function

Line Integrals in Space. $w = G(x, y, z)$

Method of Evaluation. *If C is a smooth curve in 3-space defined by the parametric equations: $x = f(t)$, $y = g(t)$, $z= h(t)$, $a \le t \le b$, then*

$$\int_C G(x,y,z)dz = \int_a^b g(f(t), g(t), h(t))h'(t)dt.$$

The integrals $\int_C G(x,y,z)dx$ *and* $\int_C G(x,y,z)dy$ *are evaluated in a similar fashion. The line integral WRT to arc length is*

$$\int_C G(x,y,z)ds = \int_a^b g(f(t), g(t), h(t))\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt.$$

We are often concerned with line integrals in the form of a sum:

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

Example 5* (p. 494) Line Integrals on a Curve in 3-space.

Note that since $\underline{F}(x, y) \bullet dr = P(x, y)dx + Q(x, y)dy$, *where* $\underline{F}(x, y) = P(x, y)\underline{i} + Q(x, y)\underline{j}$ *and* $d\underline{r} = dx\underline{i} + dy\underline{j}$ *then we may express the line integral more compactly as*

$$\int_C P(x, y)\,dx + Q(x, y)\,dy = \int_C \underline{F} \bullet d\underline{r}$$

Similarly, for a line integral on a 3-space curve,

$$\int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = \int_C \underline{F} \bullet d\underline{r},$$

where $\underline{F}(x, y, z) = P(x, y, z)\underline{i} + Q(x, y, z)\underline{j} + R(x, y, z)\underline{k}$ *and* $d\underline{r} = dx\underline{i} + dy\underline{j}$

Work. *The* work *done by a force* $\underline{F}$ *acting on an (massless) object moving along C (with a position vector:* $\underline{r}(t)$*) is defined as*

$$W = \int_C P(x, y)dx + Q(x, y)dy = \int_C \underline{F} \bullet d\underline{r}$$

The units of work depend on the units of $\|F\|$ *and on the units of distance.*

Example 6*. (p.495) Work Done by a Force

9.9 Independence of Path

The value of a line integral generally depends on the curve or path *between A and B. However, there are exceptions, in other words, there are line integrals that are independent of the path between A and B.*
The following concepts will be needed in the ensuing discussion:

Differential – Functions of Two Variables. *The differential of a function* $f(x,y)$ *is*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\,.$$

A differential expression $P(x,y)dx + Q(x,y)dy$ *is said to be an* exact differential *if there exists a function* $j(x,y)$ *such that*

$$df = P(x,y)dx + Q(x,y)dy.$$

For example, the expression $x^2y^3dx + x^3y^2dy$ *is an exact differential since it is the differential of* $j(x,y) = \frac{1}{3}x^3y^3$. *(Verify this!). On the other hand* $(2y^2 - 2y)dx + (2x^2 - xy)dy$ *is not, or is it?*

Differential – Functions of Three Variables. *The differential of a function* $f(x,y,z)$ *is*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\,.$$

A differential expression $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ *is said to be an* exact differential *if there exists a function* $j(x,y,z)$ *such that*

$$df = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz\,.$$

Path Independence. *A line integral whose value is the same for <u>every</u> curve or path connecting A to B is said to be* independent of the path.

Example 1*. (p.499) An Integral Independent of Path in the Plane

9.6 THEOREM. Fundamental Theorem for Line Integrals.

Suppose there exists a function $f(x,y)$ *such that* $df = P(x,y)dx + Q(x,y)dy$*: that is,* $P(x,y)dx + Q(x,y)dy$ *is an exact differential. Then* $\int_C P(x,y)dx + Q(x,y)dy$ *depends on only the end points A and B of the path C and*

$$\int_C P(x,y)dx + Q(x,y)dy = f(B) - f(A).$$

Proof.

Notation. A line integral $\int_C P(x,y)dx + Q(x,y)dy$, *which is independent of the path between the end points A and B, is often written as*

$$\int_A^B Pdx + Qdy.$$

Example 2*. (p.500) Using Theorem 9.8

9.9 THEOREM. Test for Path Independence in the Plane.

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C P(x,y)dx + Q(x,y)dy$ *is independent of the path C if and only if* $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *for all (x, y) in the region.*

Example 3*. (p.501) A Path-Dependent integral

Example 4*. (p.501) An integral independent of the Path

Assignment. Extract the definitions of the following terms: gradient field, potential function *and* conservative force field.

9.10 THEOREM. Test for Path Independence in Space.

Let P, Q and R have continuous first partial derivatives in an open simply connected region of space. Then $\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ *is independent of the path C if and only if* $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ *and* $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ *for all (x, y, z) in the region.*

Example 6*. (p.503) An integral independent of the Path

9.12 Green's Theorem

Introduction. *One of the most important theorems in vector integral calculus relates a line integral around a piecewise smooth simple closed curve to a double integral over the region R bounded by the curve.*

Line Integrals Along Simple Closed Curves. *We say that the* positive direction *around a simple closed curve C is that direction a point on the curve must move, or the direction a person must walk on C, in order to keep the region R bounded by C to the left. See the Figures on page 519. Note also the notations for line integrals in the positive and negative directions, respectively.*

9.13 THEOREM. Green's Theorem in the Plane.

Suppose that C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\partial P/\partial y$, *and* $\partial Q/\partial x$ *are continuous on R, then*

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Proof. Read a partial proof on page 519.

Example 1*. (p.520) Using Green's Theorem

Example 3*. (p.521) Work done by a Force

Example 4*. (p.521) Green's Theorem Not applicable

9.13 Surface Integrals

Introduction. *In the xy-plane, the length of an arc of the graph of y = f(x) from x = a to x = b is given by the definite integral*

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The corresponding problem in 3-space is to find the area A(s) of that portion of the surface S given by a function z = f(x, y) having continuous first partial derivatives on a closed region R in the xy-plane. Such a surface is said to be smooth.

9.11 Definition. Surface Area

Let f be a function for which the first partial derivatives f_x and f_y are continuous on a closed region R. Then the are of the surface over R is given by

$$A(S) = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

Example 1*. (p.526) Surface Area.

Differential of Surface Area *The function*

$$dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

is called the differential of the surface area.

Evaluating Surface Integrals *Let $w = G(x, y, z)$ and $z = f(x, y)$. If G, f, f_x, and f_y are continuous throughout a region containing S then*

$$\iint_S G(x,y,z)\,dS = \iint_R G(x,y,f(x,y))\sqrt{1+[f_x]^2+[f_y]^2}\,dA.$$

Note that when G = 1 the above formula reduces to that for surface area.

Projection of S Into Other Planes *If $y = g(x, z)$ is the equation of the surface S that projects onto a region R of the xz-plane, then*

$$\iint_S G(x,y,z)\,dS = \iint_R G(x,g(x,z),z)\sqrt{1+[g_x]^2+[g_z]^2}\,dA.$$

Similarly, if $x = h(y, z)$ is the equation of the surface S that projects onto a region R of the yz-plane, then

$$\iint_S G(x,y,z)\,dS = \iint_R G(h(y,z),y,z)\sqrt{1+[h_y]^2+[h_z]^2}\,dA.$$

Mass of a Surface *Suppose $r(x,y,z)$ represents the density of a surface at any point, or mass per unit surface area; then* mass *m of the surface is given by*

$$m = \iint_S r(x,y,z)\,dS$$

Example 2*. (p.527) Mass of a Surface.

9.14 Stokes' Theorem

Introduction. *Green's Theorem in the previous section (Section 9.12) has two vector forms. In this section and the next (Section 9.16) we shall generalize these forms to three dimensions*

Vector Form of Green's Theorem. *If* $\underline{F}(x, y) = P(x, y)\,\underline{i} + Q(x, y)\,\underline{j}$ *is a 2-space vector field, then*

$$Curl\,\underline{F} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underline{k}$$

From (12) and (13) on page 495, Green's Theorem can be written in vector notation as

$$\oint_C \underline{F} \bullet d\underline{r} = \oint_C \underline{F} \bullet \underline{T}\,ds = \iint_R (Curl\,\underline{F} \bullet \underline{k})\,dA,$$

that is, the line integral of the tangential component of $\underline{F}$ *is the double integral of the normal component of Curl* $\underline{F}$.

9.14 THEOREM. Stokes' Theorem (Green's Theorem in 3-Space).
Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve C. Let

$$\underline{F}(x, y, z) = P(x, y, z)\,\underline{i} + Q(x, y, z)\,\underline{j} + R(x, y, z)\,\underline{k}$$

Be a vector field for which P, Q and R are continuous and have continuous first partial derivatives in a region of 3-space containing S. If C is traversed in the positive direction, then

$$\oint_C \underline{F} \bullet d\underline{r} = \oint_C (\underline{F} \bullet \underline{T})\,ds = \iint_S (Curl\,\underline{F}) \bullet \underline{n}\,dS,$$

where $\underline{n}$ *is a unit normal to S in the direction of the orientation of S.*

Example 1*. (pp.534-6) Verifying Stokes' Theorem

Example 2*. (pp.536) Using Stokes' Theorem

9.16 Divergence Theorem

Introduction. *In this section we present a second form of Green's Theorem.*

Another Vector Form of Green's Theorem. *Let* $\underline{F}(x, y) = P(x, y)\,\underline{i} + Q(x, y)\,\underline{j}$ *be a 2-space vector field, and let* $\underline{T} = (dx/ds)\,\underline{i} + (dy/ds)\,\underline{j}$ *be a unit tangent to a simple closed plane curve C. In Section 9.4 we saw that* $\oint_C (\underline{F} \bullet \underline{T})ds$ *can be evaluated by a double integral involving curl* $\underline{F}$. *Similarly, if* $\underline{n} = (dy/ds)\,\underline{i} - (dx/ds)\,\underline{j}$ *is a unit normal to C, then* $\oint_C (\underline{F} \bullet \underline{n})ds$ *can be expressed in terms of a double integral of div* $\underline{F}$. *From Green's Theorem,*

$$\oint_C (\underline{F} \bullet \underline{n})ds = \oint_C Pdy - Qdx = \iint_R \left[\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{dy}\right)\right]dA = \iint_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{dy}\right]dA,$$

that is,

$$\oint_C (\underline{F} \bullet \underline{n})ds = \iint_R div\underline{F}dV.$$

This result is a special case of the divergence *or* Gauss' theorem *below.*

9.15 THEOREM. Divergence Theorem (Gauss' Theorem in 3-Space).
Let D be a closed and bounded region in 3-space with a piecewise smooth boundary S that is oriented outward. Let

$$\underline{F}(x, y, z) = P(x, y, z)\,\underline{i} + Q(x, y, z)\,\underline{j} + R(x, y, z)\,\underline{k}$$

be a vector field for which P, Q and R are continuous and have continuous first partial derivatives in a region of 3-space containing D. Then

$$\iint_S (\underline{F} \bullet \underline{n})\,dS = \iiint_D div\underline{F}dV$$

where $\underline{n}$ *is a unit normal to S in the direction of the orientation of S.*

Proof. Read a partial proof on pages 550-2.

Example 1*. (pp.552-3) Verifying Divergence Theorem

Example 2*. (p.553) Using Divergence Theorem