4.1 Definition of the Laplace Transform

Introduction. *In elementary calculus you learned that differentiation and integration are* transforms — *this means, roughly speaking, that these operations transform a function into another function. The Laplace transform has many interesting properties like linearity, that make it very useful in solving linear initial-value problems.*
If f(x, y) is a function of two variables, then a definite integral of f WRT to one of the variables leads to a function of the other variable. For example, by holding y constant we see that $\int_1^2 2xy^2 dx = 3y^2$. *Similarly, a definite integral such as* $\int_a^b K(s,t)\, f(t) dt$ *transforms a function f(t) into a function of the variable s. We are particularly interested in integral transforms of this last kind, where the interval of integration is the unbounded interval* $[0, \infty)$.
Basic definition. *If f(t) is defined for* $t \geq 0$, *then the improper integral* $\int_0^\infty K(s,t)\, f(t) dt$ *is defined as a limit:*

$$\int_0^\infty K(s,t)\, f(t) dt = \lim_{b\to\infty} \int_0^b K(s,t)\, f(t) dt.$$

If the limit exists, the integral is said to exist or to be convergent; if the limit does not exist, the integral does not exist and is said to be divergent.

4.1 DEFINITION Laplace Transform
Let f(t) be a function defined for $t \geq 0$. *Then the integral*

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$$

is said to be the Laplace transform *of f(t), provided the integral converges.*

Examples 1-4 (pp.195-6).
The following result from MATH 102 is used in evaluating Laplace transforms: For all s, n > 0, $\lim_{t\to\infty} t^n e^{-st} = 0$. *This result is proved by repeated application of L'Hopital's Rule*

$\mathcal{L}$ is a Linear Transform. $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, *where a and b are constants.*

4.1 THEOREM Transforms of Some Basic Functions

(a) $\mathcal{L}\{1\} = \frac{1}{s}$

(b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n= 1, 2, 3, …

(c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

(d) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

(e) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

(f) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$

(g) $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

4.2 The Inverse Transform and Transforms of Derivatives

Introduction. *In this section we discuss the concept of* inverse *Laplace transform and examine the transform of derivatives we then use the Laplace transform to solve some simple ordinary differential equations* (ODE's).

4.2.1 Inverse Transforms

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the inverse Laplace transform *of $F(s)$ and write $\mathcal{L}^{-1}\{F(s)\} = f(t)$. For example, from Examples 1-3 in Section 4.1 we have, respectively,*

$\mathcal{L}^{-1}\{1/s\} = 1$, $\mathcal{L}^{-1}\{1/s^2\} = t$ *and* $\mathcal{L}^{-1}\{1/(s+3)\} = e^{-3t}$

4.3 THEOREM Some Inverse Transforms

$$\text{(a) } \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

(b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ n = 1, 2, 3, …

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(d) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

(e) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

(f) $\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$

(g) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

Examples 1-3 (pp.200-1).

4.2.2 Transforms of Derivatives

As pointed out in the introduction to this chapter, our immediate goal is to use the Laplace transform to solve differential equations. To that end we need to evaluate quantities such as $\mathcal{L}\{dy/dt\}$ *and* $\mathcal{L}\{d^2y/dt^2\}$*. For example, if* $f'(t)$ *is continuous for* $t \geq 0$*, then integration by parts gives*

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

or $\quad \mathcal{L}\{f'(t)\} = sF(s) - f(0).$

Note that we have used the fact that $e^{-st} f(t) \to 0$ *as* $t \to \infty$*. Similarly, using the above result and integration by parts we deduce*

4.4 THEOREM Transform of Derivatives

If f, f', f'', … , $f^{(n-1)}$ are continuous on $[0, \infty]$ *and are of exponential order and if ,* $f^{(n)}(t)$ *is piece-wise continuous on* $[0, \infty]$*, then*

$$\mathcal{L}\{f'(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots - f^{(n-1)}(0)$$

where $\mathcal{L}\{f(t)\} = F(s)$.

Solving Linear ODEs. *It is apparent from the general result given in Theorem 4.4 that* $\mathcal{L}\{d^n y/dt^n\}$ *depends on* $Y(s) = \mathcal{L}\{y(t)\}$ *and also on the* n – 1 *derivatives of y(t) evaluated at* t = 0. *This property makes the Laplace transform ideally suited for solving DEs of the form:*

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t),$$

$$y(0) = y_0,\ y'(0) = y_1,\ \cdots,\ y^{(n-1)}(0) = y_{n-1},$$

where the a_i, $i = 0, 1, 2, \ldots, n$ *and* $y_0, y_1, \ldots, y_{n-1}$ *are constants.*

Example 4* (p.203). Solving a First-Order IVP

Example 5* (p.204). Solving a Second-Order IVP

4.3 Translation Theorems

Introduction. *In this section we present some several labour-saving theorems that enable us to build up a more extensive list of transforms (see the table in Appendix III on pages APP6-8) without the necessity of using the definition of the Laplace transform.*

4.3.1 Translation on the s-axis.

4.6 THEOREM First Translation Theorem or First Shifting Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ *and a is any real number, then*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

[*Equivalently, we may write* $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a}$.]

Example 2* (pp.208-9). Partial Fractions and Completing the Square

Example 4* (p.210). An IVP

4.3.2 Translation on the t-axis.

Unit Step Function *In engineering, one frequently encounters functions that are either "off" or "on". For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. It is convenient, then, to define a special function that is the number 0 (OFF) up to a certain time t = a and the number 1 (ON) after that time. This function is called the* unit step function *or the* Heaviside function.

4.3 DEFINITION Unit Step Function

The unit step function $\mathcal{U}(t-a)$ *is defined to be*

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

Using the definition of the Laplace transform we see that

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^a 0e^{-st}\,dt + \int_a^\infty e^{-st}\,dt = \frac{e^{-sa}}{s}.$$

A general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)\,\mathcal{U}(t-a) + h(t)\,\mathcal{U}(t-a),$$

while a function of the type

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$

is the same as

$$f(t) = g(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)].$$

Example 5* (pp.211-2). A Piecewise-Defined Function

For a general function y = f(t) consider the piecewise-defined function

$$f(t-a)\,\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a) & t \ge a \end{cases}$$

which plays a significant role in the discussion that follows.

4.7 THEOREM Second Translation Theorem or Second Shifting Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ *and* $a > 0$, *then*

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

[Inverse form: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$.].

Proof. Read the proof on p. 212

Example 6* (p.212). Using Inverse Form of Theorem 4.7

Alternative Form of Theorem 4.7.

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

Example 7* (p.213). Second Translation Theorem— Alternative Form

4.4 Additional Operational Properties

4.4.1 Derivatives of Transforms. *The Laplace transform of the product of a function f(t) with t can be found by differentiating the Laplace of f(t). If F(s) = ℒ{f(t)} and if we assume that interchanging of differentiation and integration is possible, then*

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]dt = -\int_0^\infty e^{-st} tf(t)dt = -\mathcal{L}\{tf(t)\};$$

that is,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}.$$

Similarly,

$$\mathcal{L}\{t^2 f(t)\} = \mathcal{L}\{t.tf(t)\} = -\frac{d}{ds}\mathcal{L}\{tf(t)\}. = -\frac{d}{ds}\left(-\frac{d}{ds}\mathcal{L}\{f(t)\}\right).$$

$$= \frac{d^2}{ds^2}\mathcal{L}\{f(t)\}.$$

These two cases suggest the following result which can be proved by induction.

4.8 THEOREM Derivatives of transforms
If F(s) = ℒ{f(t)} and n = 1, 2, 3, …, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s).$$

Example 1* (p.219). Using Theorem 4.8

Example 2* (pp.219-20). An Initial-Value Problem

4.42 Transforms of Integrals.

Convolution *If functions f and g are piecewise continuous on* $[0, \infty)$ *then a special product, denoted by* $f * g$, *is defined by the integral*

$$f * g = \int_0^t f(t)g(t - t)dt = H(t)$$

and is called the convolution of f and g.

Remark. *It can be shown that* $f * g = g * f$.

4.9 THEOREM Convolution Theorem

If functions f and g are piecewise continuous on $[0, \infty)$ *and of exponential order, then*

$$\mathcal{L}(f * g) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s).$$

Read the proof on p.220.

Example 3* (p.221). Transform of a Convolution

Inverse Form of Theorem 4.9: $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$.

Example 4* (pp.221). Inverse Transform as a Convolution

Transform of an Integral *When g(t) = 1 and* $\mathcal{L}\{g(t)\} = G(s) = 1/s$, *the convolution theorem implies that the Laplace transform of the integral of f is*

$$\mathcal{L}\{\int_0^t f(t)dt\} = \mathcal{L}\{1 * f(t)\} = G(s)F(s) = \frac{F(s)}{s}.$$

And the inverse form:

$$\int_0^\infty f(t)dt = \mathcal{L}^{-1}\{\frac{F(s)}{s}\}$$

can be used in lieu of partial fractions when s^n *is a factor of the denominator and* $f(t) = \mathcal{L}^{-1}\{F(s)\}$ *is easy to integrate. For example, we know for f(t) = sin t that* $F(s) = \frac{1}{s^2+1}$, *and so*

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \int_0^t \sin t\, dt = [-\cos t]\Big|_0^t = 1 - \cos t$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s.\{s(s^2+1)\}}\}$$

$$= \int_0^t (1-\cos t)\, dt = [t - \sin t]\Big|_0^t = t - \sin t$$

Ex. Show that

$$\mathcal{L}^{-1}\{\frac{1}{s^3(s^2+1)}\} = \frac{t^2}{2} - 1 + \cos t \text{ and } \mathcal{L}^{-1}\{\frac{1}{s^4(s^2+1)}\} = \frac{t^3}{3!} - t + \sin t.$$

Guess a formula for

$$\mathcal{L}^{-1}\{\frac{1}{s^{2n}(s^2+1)}\} \text{ and } \mathcal{L}^{-1}\{\frac{1}{s^{2n+1}(s^2+1)}\}, n = 0, 1, 2, \ldots$$

Volterra Integral Equation *The Volterra integral equation for f(t) is*

$$f(t) = g(t) + \int_0^t f(t)h(t-t)dt$$

where g(t) and H9t) are known.

Example 5. (p. 222) An Integral Equation

Transform of a Periodic Function *If a periodic function has period T, T > 0, then f(t + T) = f(t). The Laplace transform of a periodic function can be obtained by integrating over one period.*

4.10 THEOREM Transform of a Periodic Function

If f(t) is piecewise continuous on $[0, \infty)$ *and of exponential order, and periodic with period T, then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt.$$

Proof. (p. 224)

Example 7*. (p. 224/ #49 p. 226) Transform of a Periodic Function