12.1 Orthogonal Functions

Introduction *In this section we shall see how the two vector concepts of inner or dot product and orthogonality of vectors can be extended to functions*

Inner Product *Recall that if* u = u_1 i + u_2 j + u_3 k *and* v = v_1 i + v_2 j + v_3 k *then* u • v = $u_1v_1 + u_2v_2 + u_3v_3$.

The inner product of functions is defined below

12.1 DEFINITION Inner Product of Functions

The inner product *of two functions* f_1 *and* f_2 *on the interval [a, b] is the number*

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)dx.$$

12.2 DEFINITION Orthogonal Functions

Two functions f_1 *and* f_2 *are said to be* orthogonal *on the interval [a, b] if*

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)dx = 0.$$

For example, $f_1 = 2x^2$ *and* $f_2 = x^3/3$ *are orthogonal on [–3, 3] since*

$$(f_1, f_2) = \tfrac{2}{3}\int_{-2}^{2} x^2 x^3 dx = \tfrac{2}{3}\int_{-2}^{2} x^5 dx = \tfrac{1}{9}x^6 \Big|_{-2}^{2} = 0$$

12.3 DEFINITION Orthogonal Sets

A set of real-valued functions $\{f_0, f_1, f_2, \ldots\}$ *is said to be* orthogonal *on the interval [a, b] if*

$$(f_m, f_n) = \int_a^b f_m(x) f_n(x)dx = 0,\ m \neq n.$$

Example 1* (p. 654). Orthogonal Set of Functions

Determine which of these sets: {1, sinh x, sinh 2x, …} and {sin x, sin 3x, sin 5x, …} is (are) orthogonal on the interval [–3, 3].

Orthonormal Sets *Note that for a vector* $\underline{u}$

$$(\underline{u}, \underline{u}) = u_1^2 + u_2^2 + u_3^2 = \|\underline{u}\|^2$$

and so

$$\|\underline{u}\| = \sqrt{(\underline{u}, \underline{u})}\ .$$

Similarly, we define

$$\|f_n(x)\|^2 = (f_n, f_n) = \int_a^b f_n^2(x)dx$$

and so

$$\|f_n(x)\| = \sqrt{(f_n, f_n)} = \sqrt{\int_a^b f_n^2(x)dx}\,.$$

If $\{f_n(x)\}$ *is an orthogonal set of functions on an interval [a, b] with the additional property that:* $\|f_n(x)\| = 1$ *for all n = 0, 1, 2, then* $\{f_n(x)\}$ *is an orthonormal set on the interval.*

Note that if $\{f_0, f_1, f_2, \ldots\}$ *is orthogonal then* $\{\frac{f_0}{\|f_0\|}, \frac{f_1}{\|f_1\|}, \frac{f_2}{\|f_2\|}, \ldots\}$ *is always orthonormal.*

Example 2* (p. 654). Norms

Find the norms of all the functions in Example 1.*

Orthogonal Series Expansion *Suppose* $\{f_n(x)\}$ *is an infinite orthogonal set of functions on an interval [a, b]. We ask: If y = f(x) is a function defined on the interval [a, b], is it possible to determine a set of coefficients* c_n*, n = 0, 1 , 2, …, for which*

$$f(x) = c_0 f_0(x) + c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) + \ldots? \qquad (1)$$

Yes, we can find the coefficients by utilizing the inner product.

12.4 DEFINITION Orthogonal Set/Weight Function

A set of real-valued functions $\{f_0, f_1, f_2, \ldots\}$ *is said to be* orthogonal with respect to a weight function *w(x) on an interval [a, b] if*

$$\int_a^b w(x) f_m(x) f_n(x) dx = 0, \; m \neq n.$$

Usually we assume: w(x) > 0 on the interval of orthogonality [a, b].

The series (2) is said to be an orthogonal series expansion of *f or a* generalized Fourier series.

12.2 Fourier Series

Introduction *In this section we shall learn how to expand functions in terms of a special orthogonal set of trigonometric functions.*

Trigonometric Series

12.5 DEFINITION Fourier Series

The Fourier Series *of a function f defined on the interval (-p, p) is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{np}{p}x + b_n \sin\frac{np}{p}x \right),$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{np}{p}x\,dx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{np}{p}x\,dx.$$

The coefficients a_n, b_n *are called* Fourier coefficients *of f.*

Example 1* (p. 660). (#5, p. 662) Expansion in a Fourier Series

Convergence of a Fourier Series *The following theorem gives sufficient conditions for convergence of a Fourier series at a point.*

12.1 THEOREM Conditions for Convergence

Let f and f' be piecewise continuous on the interval (–p, p); that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to f(x) at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where f(x+) and f(x–) denote the limit of f at x from the right and from the left respectively.

Example 2 (p. 660). Convergence of a Point of Discontinuity

Periodic Extension *When f is piecewise continuous and the right- and left-hand derivatives exist at x = –p and p, respectively, then its Fourier series converges to the average [f(p+) + f(p–)]/2 at these endpoints and to this value extended periodically to* $\pm 3p, \pm 5p, \pm 7p,$ *and so on.*

Problem # 17 (p. 662)

Problem # 18 (p. 662)

12.3 Fourier Cosine and Sine Series

Review *The effort expended in the evaluation of coefficients* a_0, a_n *and* b_n *in expanding a function f in a Fourier series is reduced significantly when f is either an even or an odd function.*

Even and Odd Functions *A function f is said to be* ***even*** *if f(x) = f(–x) and* odd *if f(x) = – f(–x). For example,*

12.2 THEOREM Properties of Even/Odd Functions

(a) *The product of two even functions is even.*
(b) *The product of two odd functions is even.*
(c) *The product of an even function and an odd function is even.*
(d) *The sum (difference) of two even functions is even.*
(e) *The sum (difference) of two odd functions is odd.*
(f) *If f is even, then* $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
(g) *If f is odd, then* $\int_{-a}^{a} f(x)dx = 0.$

Cosine and Sine Series *If f is an even function on (–p, p), then the coefficients a_0, a_n and b_n from Section 12.2 become*

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{np}{p}x\,dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{np}{p}x\,dx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{np}{p}x\,dx = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{np}{p}x\,dx = 0.$$

Similarly, when f is odd on the interval (–p, p),

$$a_n = 0,\ n = 0, 1, 2, \ldots, \qquad b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{np}{p}x\,dx.$$

We summarize the results in the following definition.

12.6 DEFINITION Fourier Cosine and Sine Series

(i) The Fourier Series *of an even function f defined on the interval (-p, p) is the* cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{np}{p}x,$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx,$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{np}{p}x\,dx.$$

(ii) The Fourier Series *of an odd function f defined on the interval (-p, p) is the* sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{np}{p}x,$$

where

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{np}{p}x\,dx.$$

Example 1* (pp.664-5) Expansion in a Cosine Series

Half-Range Expansions. *If* $y = f(x)$ *is defined on* $0 < x < L$ *it can still be expanded as a Fourier series by first supplying an 'arbitrary' definition on the interval* $-L < x < 0$*, that is, extending the domain of definition of the function to* $-L < x < L$*. The three most important cases are:*

- *(i)* *reflect the graph of the function about the y- axis onto* $-L < x < 0$*; the function is now even on* $-L < x < L$*; or*
- *(ii)* *reflect the graph of the function through the origin onto* $-L < x < 0$*; the function is now odd on* $-L < x < L$*; or*
- *(iii)* *define f on* $-L < x < 0$ *by* $f(x) = f(x + L)$

The expansions obtained in this manner are called half-range expansions.

Remark. *In the first two cases the expansion is exactly as in the cosine and sine series expansion. In the last cases we have to consider the expansion on the interval* $(-L/2, L/2)$*.*

Example 3* (pp.666-7) Expansion in Three Series

Expand $f(x) = \sin x$, $0 < x < L$, *(a) in a cosine series; (b) in a sine series; and (c) in a Fourier series.*

12.4 Complex Fourier Series

Introduction *In this section we are going to use Euler's formula:*

$$e^{ix} = \cos x + i\sin x \ \textit{which implies}\ e^{-ix} = \cos x - i\sin x$$

to recast the Fourier series in Definition 12.5 into a complex form *or* exponential form.

Complex Fourier Series

12.7 DEFINITION Complex Fourier Series
The complex Fourier Series *of a function f defined on an interval (-p, p) is given by*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inpx/p},$$

$$\textit{where}\ c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-inpx/p}x\,dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

If f satisfies the hypotheses of Theorem 12.1, a complex Fourier series converges to f(x) at a point of continuity and to the average

$$\frac{f(x+) + f(x-)}{2},$$

at a point of discontinuity.

Example * (pp.671-2) Complex Fourier Series

12.5 Sturm-Liouville Theorem

Review

Some Ordinary Differential Equations (ODEs)

Linear DEs	General Solution
$y'+ay=0,$	$y=c_1e^{-ax}$
$y''+a^2y=0, \quad a>0$	$y=c_1\cos ax+c_2\sin ax$
$y''-a^2y=0, \quad a>0$	$\begin{cases} y=c_1e^{-ax}+c_2e^{ax}, \textit{ or} \\ y=c_1\cosh ax+c_2\sinh ax \end{cases}$

Cauchy-Euler DEs	General Solution *x > 0*
$x^2y''+xy'-a^2y=0,$	$\begin{cases} y=c_1x^{-a}+c_2x^{a}, \textit{ or} \\ y=c_1+c_2\ln x \end{cases}$

Parametric Bessel DEs	General Solution *x > 0*
$x^2y''+xy'+a^2y=0, \quad a\ge 0$	$y=c_1J_0(ax)+c_2Y_0(ax)$

Legendre's DE (n = 0, 1, 2, …)	Particular Solutions are polynomials
$(1-x^2)y''-2xy'+n(n+1)y=0,$	$y=P_0(x)=1,$ $y=P_1(x)=x,$ $y=P_2(x)=\frac{1}{2}(3x^2-1),\ldots$

Eigenvalues and Eigenfunctions *An orthogonal set of functions can be generated by solving a two-point boundary-value problem (BVP) involving a second-order DE containing a parameter say,* l.

Example 2 (pp. 168-9) Nontrivial Solutions of a BVP
Solve the BVP: $y''+ly=0,$ *y(0) = 0, y(L) = 0.*

Solution. *We consider THREE cases:* $l=0,\ l<0$ *and* $l>0.$

Case I. *For* $\lambda = 0$, *the general solution of the DE is* $y = c_1 x + c_2$. *Applying the boundary conditions in turn yields* $c_1 = 0 = c_2$. *Hence we only have the trivial solution:* $y = 0$.

Case II. *For* $\lambda < 0$, *it is convenient to write* $\lambda = -\alpha^2$, *where* $\alpha > 0$. *Hence the auxiliary equation is* $m^2 - \alpha^2 = 0$, *which has roots* $m_1 = \alpha$ *and* $m_2 = -\alpha$. *Since the interval under consideration is [0, L], we choose the hyperbolic expression for the general solution of the DE, i. e.,* $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$.. *Applying the boundary conditions in turn yields* $c_1 = 0 = c_2$. *Once again, we only have the trivial solution:* $y = 0$.

Case III. *For* $\lambda > 0$, *it is convenient to write* $\lambda = \alpha^2$, *where* $\alpha > 0$. *The auxiliary equation is* $m^2 + \alpha^2 = 0$, *which has complex roots* $m_1 = i\alpha$ *and* $m_2 = -i\alpha$ *and so the general solution of the DE is* $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. *As before* $y(0) = 0$ *yields* $c_1 = 0$ *and so* $y = c_2 \sin \alpha x$. *Next* $y(L) = 0$ *implies* $c_2 \sin \alpha L = 0$. *If* $c_2 = 0$ *then necessarily* $y = 0$. *But this time we can require* $c_2 \neq 0$ *since* $\sin \alpha L = 0$ *is satisfied by any integer multiple of* π.

$$\alpha L = n\pi \;\; or \;\; \alpha = \frac{n\pi}{L} \;\; or \;\; \lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, 3, \ldots$$

Therefore for any real nonzero c_2, $y = c_2 \sin(n\pi x / L)$ *is a solution of the DE for each n. since the DE is homogeneous, any constant multiple of a solution is also a solution. Thus we may, if desired, simply take* $c_2 = 1$. *In other words, for each number in the sequence*

$$\lambda_1 = \frac{\pi^2}{L^2}, \; \lambda_2 = \frac{4\pi^2}{L^2}, \; \lambda_3 = \frac{9\pi^2}{L^2}, \ldots$$

the corresponding *function in the sequence*

$$y_1 = \sin \frac{\pi}{L} x, \; y_2 = \sin \frac{2\pi}{L} x, \; y_3 = \sin \frac{3\pi}{L} x, \ldots$$

is a nontrivial solution of the original problem.

Definitions *The numbers* $\lambda_n = n^2 \pi^2 / L^2$, $n = 1, 2, 3, \ldots$ *for which the BVP above has a nontrivial solution are known as* characteristic values *or, more commonly,* eigenvalues. *The solutions depending on theses values of* λ_n, $y_n = c_2 \sin(n\pi x / L)$ *or simply* $y_n = \sin(n\pi x / L)$ *are called* characteristic functions *or, more commonly,* eigenfunctions.

Example 1 (p. 675) Eigenvalues and Eigenfunctions
Solve the BVP: $y'' + l y = 0,\ y'(0) = 0,\ y'(L) = 0.$

Regular Sturm-Liouville Problem *Let p, q, r and r' be real-valued functions continuous on an interval [a, b], and let r(x) > 0 and p(x) > 0 for every x in the interval. Then*

Solve: $$\frac{d}{dx}[r(x)y'] + (q(x) + l p(x))y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(a) + B_2 y'(a) = 0$$

is said to be a regular Sturm-Liouville problem. *The coefficients in the boundary conditions are assumed to be real and independent of* l.

12.3 THEOREM Properties of the Regular Sturm-Liouville Problem
(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $l_1 < l_2 < l_3 < \cdots < l_n \cdots$ *such that* $l_n \to \infty$ *as* $n \to \infty$.
(b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).
(c) Eigenfunctions corresponding to different eigenvalues are linearly independent.
(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function p(x) on the interval [a, b],
i. e., $\int_a^b p(x) y_m y_n dx = 0, \quad m \neq n.$

Read the proof of (d) on page 676.

Example 2* (p.677) A Regular Sturm-Liouville Problem

Self-Adjoint Form *If we carry out the differentiation* $\frac{d}{dx}[r(x)y']$, *the* self-adjoint *differential equation*

$$\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \qquad (1)$$

becomes

$$r(x)y'' + r'(x)y' + (q(x) + \lambda p(x))y = 0.$$

For example, Legendre's *DE*: $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ *is of the above form with* $r(x) = 1 - x^2$ *and so* $r'(x) = -2x$. *In other words, another way of writing Legendre's DE is*

$$\frac{d}{dx}[(1 - x^2)y'] + n(n+1)y = 0.$$

Moreover, Bessel's *equation,* Cauchy-Euler *equations and* DE's with constant coefficients *all have the above form. In fact, if the coefficients are continuous and* $a(x) \neq 0$ *for all x in some interval, then any second order DE*

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

can be recast into the so-called self-adjoint form. Read how to do this from pages 678– 9.

Remark. We use the form in (1) *to determine the weight function p(x) needed in the orthogonality relation.*

Bessel Equation: $x^2y''+xy'+(x^2-n^2)y=0,$ *where* n *is an integer has solution on* $(0,+\infty)$

$$J_n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+n)!}(x/2)^{2n+n} \qquad (2)$$

called Bessel's function of the first kind of order n . *Letting* $t=ax, a>0$ *in*

$$x^2y''+xy'+(a^2x^2-n^2)y=0, \qquad (3)$$

then by the Chain Rule

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=a\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=a^2\frac{d^2y}{dt^2}.$$

Accordingly, (3) becomes

$$\left(\frac{t}{a}\right)^2a^2\frac{d^2y}{dt^2}+\left(\frac{t}{a}\right)a\frac{dy}{dt}+(t^2-n^2)y=0$$

or

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-n^2)y=0.$$

Example 3 (p.677) Parametric Bessel DE

Example 3* (p.677) Laguerre's and Hermite's DEs

12.6 Bessel and Legendre Series

12.6.1 Fourier-Bessel Series *We saw in Example 3 of Section 12.5 that for a fixed value of n the set of Bessel functions* $\{J_n(a_i x)\}$, $i = 1, 2, 3, \ldots$ *is orthogonal WRT the weight function p(x) = x on an interval* $[0, b]$ *when the* $a_i s$ *are defined by means of a boundary condition of the form*

$$A_2 J_n(ab) + B_2 a J_n'(ab) = 0. \qquad (1)$$

The eigenvalues of the corresponding Sturm-Liouville problem are $l_i = a_i^2$. *The orthogonal series expansion or generalized Fourier series of a function f defined on the interval* $(0, b)$ *(see Section* 12.1*) in terms of this orthogonal set is*

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(a_i x),$$

where

$$c_i = \frac{\int_0^b x J_n(a_i x) f(x) dx}{\| J_n(a_i x) \|^2}.$$

This is known as Fourier-Bessel series. *The square norm of the function* $J_n(a_i x)$ *is defined by (see Section 12.1)*

$$\| J_n(a_i x) \|^2 = \int_0^b x J_n^2(a_i x) dx$$

Differential Recurrence Relations *The following differential recurrence relations are often useful in the evaluation of the coefficients* c_i:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

12.8 DEFINITION Fourier-Bessel Series

The Fourier-Bessel Series *of a function f defined on the interval (0, b) is given by*

(i)
$$f(x)=\sum_{i=1}^{\infty} c_i J_n(a_i x),$$
$$c_i=\frac{2}{b^2 J_{n+1}^2(a_i b)}\int_0^b xJ_n(a_i x)\,f(x)dx$$

where the a_i*s are defined by* $J_n(ab)=0$.

(ii)
$$f(x)=\sum_{i=1}^{\infty} c_i J_n(a_i x),$$
$$c_i=\frac{2a_i^2}{a_i^2 b^2-n^2+h^2 J_n^2(a_i b)}\int_0^b xJ_n(a_i x)\,f(x)dx$$

where the a_i*s are defined by* $hJ_n(ab)+abJ_n'(ab)=0$.

(iii)
$$f(x)=\sum_{i=1}^{\infty} c_i J_n(a_i x),$$
$$c_1=\frac{2}{b^2}\int_0^b xf(x)dx,\quad c_i=\frac{2}{b^2 J_0^2(a_i b)}\int_0^b xJ_0(a_i x)\,f(x)dx$$

where the a_i*s are defined by* $J_0'(ab)=0$.

12.4 THEOREM Conditions for Convergence

If f and f' be piecewise continuous on the open interval (0, b), then the Fourier-Bessel expansion of f on the interval converges to f(x) at any point of continuity. At a point of discontinuity it converges to the average
$$\frac{f(x+)+f(x-)}{2}.$$

Example 1* (p.684) Expansion in a Fourier-Bessel Series

Example 2* (p.685) Expansion in a Fourier-Bessel Series