Trigonometric Functions.

It can be shown (using the limit definition of a derivative in 12.1) that

$$\frac{d}{dx}[\sin x] = \cos x \quad \text{and} \quad \frac{d}{dx}[\cos x] = -\sin x. \qquad (1)\&(2)$$

Moreover, by the quotient rule

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \quad \left(\text{since } \cos^2 x + \sin^2 x = 1 \ \& \ \sec x = \frac{1}{\cos x}\right). \qquad \text{- - - - -}(3)$$

Similarly, it can be shown that:

$$\frac{d}{dx}[\cot x] = -\csc^2 x, \quad \frac{d}{dx}[\sec x] = \sec x \cdot \tan x \quad \dots \quad (4)\&(5)$$

$$\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x. \quad \dots\dots \quad (6)$$

<u>Example</u> If $y = \cos(2x^2-1) + e^{\cos x}$ then

$$y' = -\sin(2x^2-1)\cdot 4x + e^{\cos x}(-\sin x) \quad \text{[Chain Rule]}$$

$$= -4x\sin(2x^2-1) - (\sin x)\cdot e^{\cos x}.$$

From the six derivative formulae above we deduce

$$\int \cos x\,dx = \sin x + c, \quad \int \sin x\,dx = -\cos x + c \qquad (7)\&(8)$$

$$\int \sec^2 x\,dx = \tan x + c, \quad \int \csc^2 x\,dx = -\cot x + c \qquad (9)\&(10)$$

$$\int \sec x \tan x\,dx = \sec x + c, \quad \int \csc x \cot x\,dx = -\csc x + c. \quad (11)\&(12)$$

<u>Examples</u>

1. $\int x^2\cos(x^3-1)\,dx = \frac{1}{3}\int \cos u\,du \quad (u = x^3-1 \Rightarrow du = 3x^2dx)$

$= \frac{1}{3}\sin u + c = \frac{1}{3}\sin(x^3-1) + c.$

2. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du \quad (u = \cos x \Rightarrow du = -\sin x\,dx)$

$= -\ln|u| + c = -\ln|\cos x| + c$

$= \ln|\sec x| + c.$

3. Show that

$$\int \cot x \, dx = \ln|\sin x| + C.$$

4. $$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)\, dx}{\sec x + \tan x}$$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C.$$

5. Show that

$$\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \tan x| + C.$$

6. $$\int x \cos x \, dx = \int x \, d(\sin x)$$

$$= x \sin x - \int \sin x \, dx \qquad \text{(Integration by Parts).}$$

$$= x \sin x + \cos x + C$$