King Fahd University of Petroleum and Minerals
Dhahran 31261
Department of Mathematical Sciences

MATH 132 (023)

Second Major Exam

July 27th, 2003

Instructor: Dr. A. Umar

Time Allowed	1 hr 15 mins
Time	7.00-8.15p.m.

SOLUTION.

Name:____________________

ID Number:____________________

Notes:
1. Students must have a valid KFUPM ID Card with them.
2. You must show all your work to justify your answer. Be as organized as possible.
3. A scratch paper is attached at the end of this question paper. **PLEASE, DO NOT REMOVE IT.**
4. Programmable Calculators and Mobile Phones are **NOT** allowed.

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Q1. Given the total-cost function $T_c = f(q)$, show that if $\frac{d}{dq}(A_c) = 0$, then the marginal-cost function and average-cost function are equal.

[15 pts]

$$T_c = f(q) = A_c \cdot q$$

Thus the marginal cost (M_c) is

$$M_c = \frac{dT_c}{dq} = \frac{d}{dq}(A_c \cdot q)$$
$$= \left(\frac{dA_c}{dq}\right)\cdot q + A_c \cdot \frac{d}{dq}(q) \quad \text{[Product Rule]}$$
$$= 0 \cdot q + A_c \cdot (1)$$
$$= A_c$$

as required.

Q2. Let C be the consumption function and I the regional income of GCC countries (1990-2000), if

$$C = \frac{10\sqrt{I} + 0.7\sqrt{I^3} - 0.2I}{\sqrt{I}} = \frac{10I^{1/2} + 0.7I^{3/2} - 0.2I}{I^{1/2}}$$

where C and I are expressed in billions of Saudi riyals. Find the GCC countries propensity to save when I = 6, 400.

[18 pts]

$$\frac{dC}{dI} = \frac{(5I^{-1/2} + 1.05I^{1/2} - 0.2)I^{1/2} - (10I^{1/2} + 0.7I^{3/2} - 0.2I)(0.5I^{-1/2})}{I}$$

$$= \frac{\cancel{5} + 1.05I - 0.2I^{1/2} - \cancel{5} - 0.35I + 0.1I^{1/2}}{I}$$

$$= \frac{0.7I - 0.1I^{1/2}}{I}$$

$$= \frac{4480 - 8}{6400} \quad (\text{when } I = 6{,}400)$$

$$= \frac{4472}{6400} \approx 0.6988.$$

Marginal Propensity to Save (M_{ps}) is

$$M_{ps} = 1 - \frac{dC}{dI} \approx 0.3012.$$

Q3. Find p' if $p = \ln\sqrt{\frac{2q+1}{2q-1}}$. $= \ln\left(\frac{2q+1}{2q-1}\right)^{1/2}$

[15 pts]

$$= \frac{1}{2}\left[\ln(2q+1) - \ln(2q-1)\right]$$

$$p' = \frac{1}{\cancel{2}}\left[\frac{1}{2q+1}(\cancel{2}) - \frac{1}{2q-1}(\cancel{2})\right]$$

$$= \frac{1}{2q+1} - \frac{1}{2q-1}$$

$$= \frac{-2}{4q^2-1} = \frac{2}{1-4q^2}.$$

Q4. For the normal-density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ ($e \approx 2.71828$), show that $f'(-\sqrt{2\pi}) = e^{-\pi}$

[15 pts]

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{d}{du}(e^u) \cdot \frac{du}{dx} \qquad \left(u = -x^2/2 \Rightarrow \frac{du}{dx} = -x\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^u \cdot (-x)$$

$$= \frac{-x}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f(-\sqrt{2\pi}) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} e^{-2\pi/2} = e^{-\pi}.$$

Q5. The GCC countries savings S from (1980-90) is defined implicitly in terms of their regional income I by the equation

$$S^2 + \tfrac{1}{4}I^2 = SI + I$$

where both S and I are billions of Saudi riyals. Find the GCC countries marginal propensity to consume when I = 16 and S = 12.

[18 pts]

$$S^2 + \frac{1}{4}I^2 = SI + I$$

(Implicit differentiation w.r.t. I)

$$2S\frac{dS}{dI} + \frac{1}{2}I = S + I\frac{dS}{dI} + 1$$

$$(2S - I)\frac{dS}{dI} = S - \frac{1}{2}I + 1$$

$$M_{PS} = \frac{dS}{dI} = \frac{2S - I + 2}{2(2S-I)} = \frac{5}{8} \quad (\text{when } I = 16,\ S = 12)$$ 9+3

Marginal Propensity to Consume (M_{PC}) is –6

$$M_{PC} = 1 - M_{PS} = 1 - \frac{5}{8} = \frac{3}{8} = 0.375 \text{ or } 37.5\%.$$

Q6. If $C = \dfrac{x(x^2+1)^2}{\sqrt{2+x^2}}$ find C'. [Hint. Use logarithmic differentiation.]

[18 pts]

$$\ln C = \ln x + \ln(x^2+1)^2 - \ln(2+x^2)^{1/2}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2+x^2)$$ 4

$$\Rightarrow \frac{1}{C}\frac{dC}{dx} = \frac{1}{x} + 2\cdot\frac{1}{x^2+1}(2x) - \frac{1}{\cancel{2}}\cdot\frac{1}{2+x^2}(\cancel{2}x)$$ 9

$$\Rightarrow \frac{dC}{dx} = C\left[\frac{1}{x} + \frac{4x}{x^2+1} - \frac{x}{2+x^2}\right]$$

$$= \frac{x(x^2+1)^2}{\sqrt{2+x^2}}\left[\frac{1}{x} + \frac{4x}{x^2+1} - \frac{x}{2+x^2}\right].$$ 5

$$= \frac{2(x^2+1)(2x^4+5x^2+1)}{(2+x^2)^{3/2}}$$

Q7. If $y = 5^{2x-3}$, find y''.

[15 pts]

$y' = 5^{2x-3}(2) \cdot \ln 5 = (2\ln 5) \cdot 5^{2x-3}$ 7

$y'' = (2\ln 5)\, 5^{2x-3}(2) \cdot \ln 5$ 7

$= (2\ln 5)^2 5^{2x-3}$ 1

Q8. If $C = 4q - q^2 + 2q^3$ is a total-cost function, when is marginal cost increasing?

[18 pts]

$M_c = C' = 4 - 2q + 6q^2$

$= 2(2 - q + 3q^2)$ 6

Now

$M_c' = 2(-1 + 6q) = -2 + 12q$ 6

Marginal Cost is increasing when $M_c' > 0$, i.e.,

$-2 + 12q > 0 \iff q > 1/6$, i.e., $q \geq 1$. 6

Q9. If $f(x) = \dfrac{x}{x^2+1}$ find its absolute extrema on [0, 2].

[15 pts]

$f(x) = x(x^2+1)^{-1}$

$f'(x) = 1(x^2+1)^{-1} + x(-1)(x^2+1)^{-2}(2x)$ 8

$= \dfrac{1}{x^2+1} - \dfrac{2x^2}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$

when $x = 1, -1$, however, only $x = 1 \in [0, 2]$

Thus

$\left.\begin{array}{l} f(0) = 0 \\ f(2) = 2/5 \end{array}\right\}$ endpoints 3

$f(1) = \frac{1}{2}$ 4

Thus $(1, 1/2)$ is an abs. max. & $(0, 0)$ is an abs. min.

Q10. For a manufacturer's product, the total-revenue function is given by

$$r = 240q + 57q^2 - q^3$$

determine the output (quantity) for maximum revenue

[18 pts]

$$r' = 240 + 114q - 3q^2$$
$$= 3(80 + 38q - q^2)$$
$$= 3(2+q)(40-q) = 0$$

8

when $q = 40$, $-\cancel{2}$ (rejected)

If $0 < q < 40$: $r' = 3(+)(+) = +$, so r is increasing 6
$q > 40$: $r' = 3(+)(-) = -$, " " " decreasing

So when $q = 40$ there is an absolute maximum since 4
it is the only rel. max on $(0, +\infty)$.
Or, since $q \geq 0$ and production level is finite, we may
consider q on $[0, a]$ for finite number $a > 40$.

Q11. Determine when $f(x) = \dfrac{x+1}{x-1}$ is concave down.

[15 pts]

$$f'(x) = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2} = -2(x-1)^{-2}$$ 5

$$f''(x) = (-2)(-2)(x-1)^{-3} = 4(x-1)^{-3} = \frac{4}{(x-1)^3}$$ 5

f is concave down when $f'' < 0$, i.e.,

$$\frac{4}{(x-1)^3} < 0 \Rightarrow (x-1)^3 < 0$$ 5

$$\Rightarrow x - 1 < 0$$

$$\Rightarrow x < 1.$$

King Fahd University of Petroleum and Minerals
Dhahran 31261
Department of Mathematical Sciences
MATH 132 (031)
Major Exam II (Make-up)
December 16th , 2003
Instructor: Dr. A. Umar

Time Allowed	1 hr 30mins.
Time	17.15-18.45 hrs.

SOLUTION

Name:______________________________

ID Number:__________________ Section: 01 or 02

Notes:

1. Students must have a valid KFUPM ID Card with them.
2. Use the cover page where appropriate.
3. You must show all your work to justify your answer. Be as organized as possible.
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ME2.031

Q1. If $f(w) = (8w^2 + 2w - 3)(5w^3 + 3)$, find df/dw.

[8 pts]

$$\frac{df}{dw} = (8w^2+2w-3)(15w^2) + (16w+2)(5w^3+3) \quad \text{(Product Rule)}$$

$$= 120w^4 + 30w^3 - 45w^2 + 80w^4 + 48w + 10w^3 + 6$$

$$= 200w^4 + 40w^3 - 45w^2 + 48w + 6.$$

Q2. Let C be the consumption function and I the regional income of GCC countries (1990-2000), if

$$C = \frac{10\sqrt{I} + 0.70125\sqrt{I^3} - 0.2I}{\sqrt{I}}$$

where C and I are expressed in billions of Saudi riyals. Find the GCC countries marginal propensity to consume when I = 6, 400.

[8 pts]

$$C = 10 + 0.70125I - 0.2\sqrt{I} \quad \text{(after simplification)}$$

$$M_{PC} = \frac{dC}{dI} = 0.70125 - 0.1/\sqrt{I}$$

$$= 0.70125 - 0.1/\sqrt{6400} \quad (\text{when } I = 6400)$$

$$= 0.70125 - 0.1/80$$

$$= 0.70125 - 1/800$$

$$= 0.70125 - 0.00125$$

$$= 0.7 \text{ or } 70\%$$

* M_{PC} is the marginal propensity to consume.

Q3. Find p' if $p = \ln\left(\sqrt[5]{\dfrac{5q+1}{5q-1}}\right)$.

[8 pts]

$$p = \frac{1}{5}\left[\ln(5q+1) - \ln(5q-1)\right] \quad \text{(after simplifications)}$$

$$p' = \frac{1}{5}\left[\frac{1}{5q+1}\cdot(5) - \frac{1}{5q-1}\cdot(5)\right]$$

$$= \frac{1}{5q+1} - \frac{1}{5q-1}$$

$$= \frac{5q-1-(5q+1)}{(5q+1)(5q-1)} = \frac{-2}{25q^2-1}.$$

Q4. For the normal-density function $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ ($e \approx 2.71828$), show that

$e^{\pi}.f'(-\sqrt{2\pi}) = 1$

[8 pts]

$$f'(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}\cdot\left(-\frac{\cancel{2}x}{\cancel{2}}\right)$$

$$= \frac{-xe^{-x^2/2}}{\sqrt{2\pi}}$$

and so

$$f'(-\sqrt{2\pi}) = \frac{-(-\sqrt{2\pi})e^{-\pi}}{\sqrt{2\pi}} = e^{-\pi}.$$

Thus

$$e^{\pi}.f'(-\sqrt{2\pi}) = e^{\pi}.e^{-\pi} = 1$$

as required.

Q5. Find all the asymptotes of

$$y = \frac{2(e^{x} + e^{-x})}{e^{x} - e^{-x}}.$$

[Hint: Multiply the numerator and denominator by e^{x} (e^{-x}) to find the limits as x approaches $-\infty$ $(+\infty)$, respectively.]

[8 pts]

Horizontal Asymptotes:

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{2(e^{x}+e^{-x})}{e^{x}-e^{-x}} = \lim_{x\to-\infty} \frac{2(e^{2x}+1)}{e^{2x}-1} \quad \left(\times \frac{e^{x}}{e^{x}}\right)$$

$$= \lim_{x\to-\infty} \frac{2e^{2x}}{e^{2x}-1} + \lim_{x\to-\infty} \frac{2}{e^{2x}-1}$$

$$= 0 - 2 = -2 \quad (\text{since } \lim_{x\to-\infty} e^{2x} = 0).$$

$$\lim_{x\to+\infty} y = \lim_{x\to+\infty} \frac{2(e^{2x}+e^{-x})}{e^{x}-e^{-x}} = \lim_{x\to+\infty} \frac{2(1+e^{-2x})}{1-e^{-2x}}$$

$$= \lim_{x\to+\infty} \frac{2}{1-e^{-2x}} + \lim \frac{e^{-2x}}{1-e^{-2x}}$$

$$= 2 + 0 = 2 \quad (\text{since } \lim_{x\to+\infty} e^{-2x} = 0)$$

Thus $y = 2$ & $y = -2$ are the H.A.'s.

VA: $x = 0$, since $e^{x} - e^{-x} = 0$ if $x = 0$.

Q6. If $C = \left(\frac{2}{x}\right)^{x}$ find C'. [Hint. Use logarithmic differentiation.]

[8 pts]

$$\ln C = \ln\left(\frac{2}{x}\right)^{x} = x\ln\left(\frac{2}{x}\right) = x\ln 2 - x\ln x$$

(implicit differentiation gives:)

$$\frac{1}{C}C' = \ln 2 - x\cdot\frac{1}{x} - \ln x$$

$$= \ln 2 - 1 - \ln x$$

$$C' = C\left[\ln(2/e) - \ln x\right]$$

$$= \left(\frac{2}{x}\right)^{x}\left[\ln(2/e) - \ln x\right].$$

Q7. If $y = 4^{2x-3}$, find $y^{(3)}$.

[8 pts]

$$y' = 4^{2x-3} \cdot 2 \cdot \ln 4 = (\ln 4^2)\, 4^{2x-3} \quad \left(= \frac{dy}{dx}\right)$$

$$y'' = \frac{d}{dx}\left[(\ln 4^2) \cdot 4^{2x-3}\right] = (\ln 4^2) \cdot \frac{d}{dx}\left[4^{2x-3}\right]$$

$$= (\ln 4^2)^2 \cdot 4^{2x-3}$$

$$y''' = (\ln 4^2)^3 \cdot 4^{2x-3}.$$

In fact,

$$y^{(n)} = (\ln 4^2)^n \cdot 4^{2x-3}.$$

Q8. If $C = 4q - q^2 + 2q^3$ is a total-cost function, when is marginal cost decreasing?

[8 pts]

$$M_c = C' = 4 - 2q + 6q^2$$

$$M_c' = -2 + 12q < 0 \Rightarrow q < 1/6$$

Thus M_c is decreasing when $q \le 0$, or for $q > 0$ M_c is always ~~increasing~~ nondecreasing.

Q9. If $f(x) = \dfrac{-x}{x^2+1}$ find its absolute extrema on $[-2, 0]$.

[8 pts]

$$f'(x) = \frac{(x^2+1)(-1) - (-x)(2x)}{(x^2+1)^2}$$

$$= \frac{-x^2 - 1 + 2x^2}{(x^2+1)^2} = \frac{x^2 - 1}{(x^2+1)^2} = \frac{(x-1)(x+1)}{(x^2+1)^2} = 0$$

$$\Rightarrow x = -1,\ 1 \text{ (rejected since } 1 \notin [-2,0]).$$

Now

$$f(0) = 0, \quad f(-1) = \frac{1}{(-1)^2+1} = \frac{1}{2}, \quad f(-2) = \frac{2}{(-2)^2+1} = \frac{2}{5}.$$

Absolute max. is $1/2$ (at $x = -1$),

Absolute min. value is 0 (at $x = 0$).

Q10. For a manufacturer's product, the total-revenue function is given by

$$T_R = 240q + 57q^2 - q^3$$

determine the maximum revenue in SR.

[8 pts]

$$T_R' = 240 + 114q - 3q^2$$
$$= 3(80 + 38q - q^2)$$
$$= 3(40 - q)(2 + q) = 0 \Rightarrow q = 40, -2 \text{ (rejected since } q \geq 0\text{)}.$$

Now

$$T_R'' = 114 - 6q$$
$$= 114 - 6(40) \quad (\text{when } q = 40)$$
$$= 114 - 240 < 0$$

Thus by the SDT, T_R has a max. value when $q = 40$.
The max. value is

$$T_R(40) = 240(40) + 57(40^2) - 40^3$$
$$= 40^2(6 + 57 - 40) = 40^2(23) = 36{,}800 \text{ SR}.$$

Q11. Determine when $f(x) = \dfrac{x-1}{x+1}$ is concave down.

[8 pts]

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2} = 2(x+1)^{-2}$$

$$f''(x) = \frac{-4}{(x+1)^3} < 0 \quad (\text{for Concave DOWN})$$

$$\Rightarrow (x+1)^3 > 0$$
$$\Rightarrow x + 1 > 0$$
$$\Rightarrow x > -1.$$

Thus $f(x) = (x-1)/(x+1)$ is concave DOWN on $(-1, +\infty)$.

Q12. Sketch the graph of a continuous function f such that f(3) = 5, both $f'(x) > 0$ and $f''(x) > 0$ for x < 3, and both $f'(x) < 0$ and $f''(x) > 0$ for x > 3. Moreover, the graph of f passes through (0, 0) and (6, 0), and has no inflection points. [12 pts]

Note that $f'(x) > 0$ and $f''(x) > 0$ for $x < 3$, means the graph of f is concave UP and all tangent lines to the graph have +ve slope on $(-\infty, 3)$, i.e, f is increasing.
Similarly, $f'(x) < 0$ and $f''(x) > 0$ for $x > 3$, means the graph of f is concave DOWN and all tangent lines to the graph have -ve slope on $(3, +\infty)$, i.e, f is decreasing.