MATH 223 REVIEW PROBLEMS

1. You are in a nicely heated cabin in the winter. Deciding that it's too warm, you open a small window. Let T be the temperature in the room, t minutes after the window was opened, x feet from the window.
   (a) Is T an increasing or decreasing function of t? Explain.
   (b) Is T an increasing or decreasing function of x? Explain.
2. Consider the following sets of points in space:
   A is the set of points (x, y, z) whose distance from the y-axis is 2.
   B is the set of points (x, y, z) whose distance from the yz-plane is 3.
   C is the set of points (x, y, z) whose distance from the z-axis is equal to its distance to the xy-plane.
   Describe in words, write equations, and give a sketch for each of these sets of points.
3. Describe in words the intersection of the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 6 - x^2 - y^2$, sketch it, and give simplified equations for it.
4. Each of the graphs in Figure 1 shows the quantity q that can be sold of a certain product (Product 1), as a function of its price p_1 and the price p_2 of another product (Product 2).

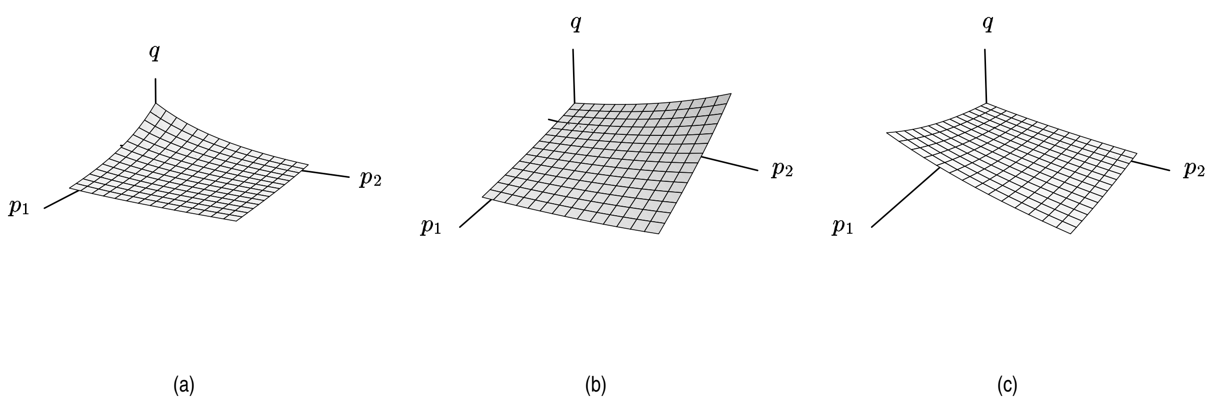


Figure 1

   Choose which graph fits best and explain why if:
   (a) Product 1 is flashlights and Product 2 is batteries.
   (b) Product 1 is domestic cars and Product 2 is imported cars.
5. Match each of the following functions in (a)-(f), given by a formula, to the corresponding table of numerical values, graph and/or contour maps (i)-(xi). Some formulas may match NONE or MORE THAN ONE of the representations (i)-(ix).

(a) $f(x,y) = x^2 - y^2$	(b) $f(x,y) = 6 - 2x + 3y$	(c) $f(x,y) = \sqrt{1 - x^2 - y^2}$
(d) $f(x,y) = \dfrac{1}{1 + x^2 + y^2}$	(e) $f(x,y) = 6 - 2x - 3y$	(f) $f(x,y) = \sqrt{x^2 + y^2}$

2

(i)

Table 1

y \ x	-2	-1	0	1	2
2	2.828	2.236	2.000	2.236	2.828
1	2.236	1.414	1.000	1.414	2.236
0	2.000	1.000	0.000	1.000	2.000
-1	2.236	1.414	1.000	1.414	2.236
-2	2.828	2.236	2.000	2.236	2.828

(ii)

Table 2

y \ x	-2	-1	0	1	2
2	0.111	0.167	0.200	0.167	0.111
1	0.167	0.333	0.500	0.333	0.167
0	0.200	0.500	1.000	0.500	0.200
-1	0.167	0.333	0.500	0.333	0.167
-2	0.111	0.167	0.200	0.167	0.111

(iii)

Table 3

y \ x	-2	-1	0	1	2
2	0.00	-3.00	-4.00	-3.00	0.00
1	3.00	0.00	-1.00	0.00	3.00
0	4.00	1.00	0.00	1.00	4.00
-1	3.00	0.00	-1.00	0.00	3.00
-2	0.00	-3.00	-4.00	-3.00	0.00

(iv)

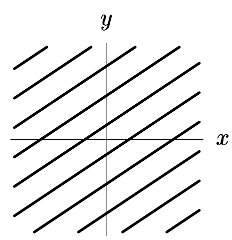


Figure 2

(v)

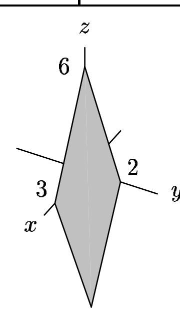


Figure 3

(vi)

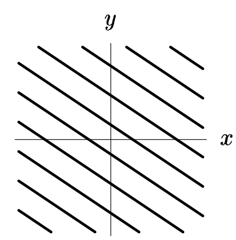


Figure 4

(vii)

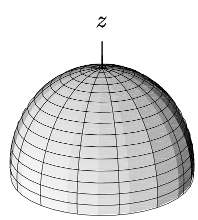


Figure 5

(viii)

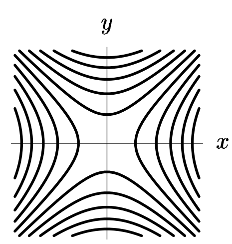


Figure 6

(ix)

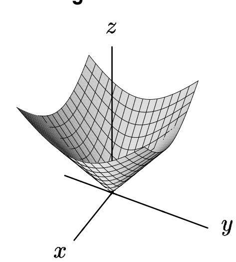


Figure 7

6. Draw a possible contour diagram for the function whose graph is shown below. Label your contours with reasonable z-values.

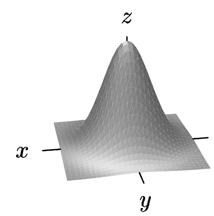


Figure 8

7. Consider the function $z = f(x, y) = y - x^2$.
   (a) Plot the level curves of the function for $z = -2, -1, 0, 1, 2$. Label them clearly.
   (b) Imagine the surface whose height above any point (x, y) is given by $z = f(x, y)$. Suppose you are standing on the surface at the point where $x = 1$, $y = 2$.
      (i) What is your height?
      (ii) If you start to move on the surface parallel to the y-axis in the direction of increasing y, does your height increase or decrease? Justify your answer using your contour map.
      (iii) Does your height increase or decrease if you start to move on the surface parallel to the x-axis in the direction of increasing x? Justify your answer using your contour map.

8. Figure 9 below shows the level curves[1] of the temperature T in degrees Celsius as a function of t hours and depth h in centimeters beneath the surface of the ground at O'Neill, Nebraska from midnight $(t = 0)$ one day to midnight $(t = 24)$ the next.

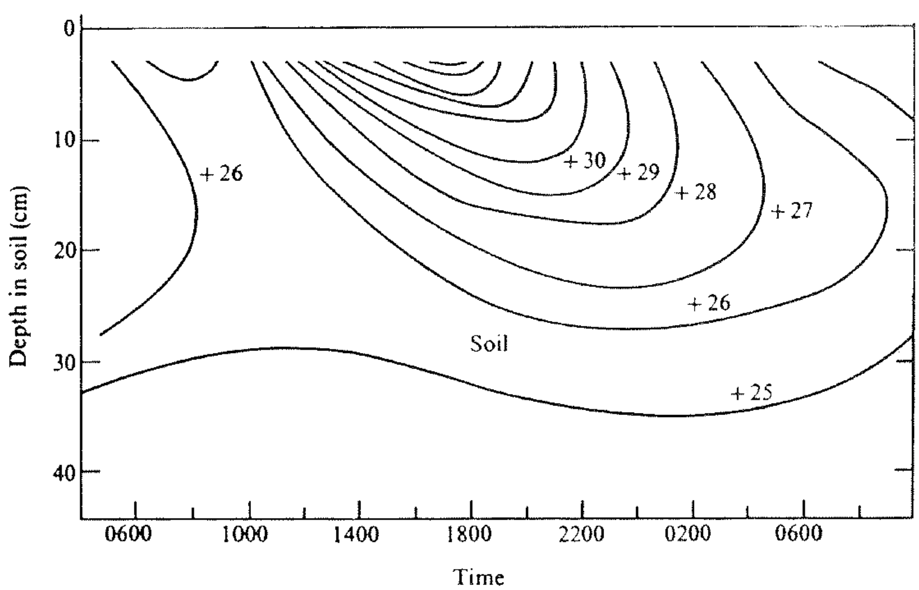


Figure 9

   (a) Approximately what time did the sun rise? When do you think the sun is directly overhead? Explain your reasoning.
   (b) Sketch graphs of the temperature as a function of time at 5 centimeters and at 20 centimeters below the surface.
   (c) Sketch a graph of the temperature as a function of the distance at noon.

[1] from S. J. Williamson, *Fundamentals of Air Pollution*, (Reading: Addison-Wesley, 1973)

9. True or False? If all of the contours of a function $f(x,y)$ are parallel lines, then the function must be linear. (If you think that the statement is true, explain why. If you think that it is false, present a counterexample.)
10. Given the table of some values of a linear function:
    (a) Complete the table:

Table 4

$y\backslash x$	2.5	3.0	3.5
-1	6		8
1		1	2
3	-6		

    (b) Determine a formula for the function.
11. True or false? Give a reason for your answer. $-\dfrac{\vec{a}}{\|\vec{a}\,\|}$ is a unit vector provided $\|\vec{a}\,\| \neq 0$.
12. If $\vec{w} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{u} = 2\vec{i} + a\vec{j} + 3\vec{k}$, $\vec{v} = \vec{i} + \vec{j}$, find:
    (a) A unit vector parallel to $\vec{w}$.
    (b) The value of a making $\vec{w}$ and $\vec{u}$ perpendicular.
    (c) A unit vector perpendicular to both $\vec{w}$ and $\vec{v}$.
13. Given the vectors $\vec{v}$ and $\vec{w}$ shown in the figure:
    (a) draw (and label) $\vec{v} + 3\vec{w}$
    (b) draw (and label) $\vec{v} - \vec{w}$

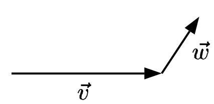


Figure 10

    (c) Is there some value of t so that $\vec{w} + t\vec{v}$ is perpendicular to $\vec{w}$? If so, estimate (by eye) the value of t (in terms of the lengths of $\vec{v}$ and $\vec{w}$). If not, explain why not.
14. Three men are trying to hold a ferocious lion still for the veterinarian. The lion, in the center, is wearing a collar with three ropes attached to it and each man has hold of a rope. Charlie is pulling in the direction N62°W with a force of 350 pounds and Sam is pulling in the direction N43°E with a force of 400 pounds. What is the direction and magnitude of the force needed on the third rope to counterbalance Sam and Charlie? Draw a diagram—label and define variables.
15. A bird is flying with velocity $\vec{v} = 10\vec{i} + 2\vec{j}$ (measured in m/sec) relative to the air. The wind is blowing at a speed of 5 m/sec parallel to the x-axis but opposing the bird's motion.
    (a) Draw a picture showing the velocity of the bird relative to the air, $\vec{v}$, the velocity of the wind, $\vec{w}$, and the velocity of the bird relative to the ground, $\vec{u}$.
    (b) Write the components of the vector $\vec{w}$.
    (c) Find the components of the vector $\vec{u}$.
    (d) Find the speed of the bird relative to the ground.

16. Say whether the following are true or false:
    (a) The angle between $-\vec{i} + 2\vec{k}$ and $\vec{i} + \vec{j} + \vec{k}$ is less than $\pi/2$.
    (b) The component of $\vec{u}$ in the direction of $\vec{v}$ is the same as the component of $\vec{v}$ in the direction of $\vec{u}$.
    (c) The two planes $z = 3x - 2y + 5$ and $6x - 2y - 2z = 0$ are parallel.
    (d) The triangle ABC with vertices $A = 2\vec{i} + 4\vec{j}$, $B = 5\vec{i} - 2\vec{j}$, and $C = -3\vec{i} - \vec{j}$ is a right triangle.
    (e) There are exactly 2 unit vectors perpendicular to the vector $\vec{i} + \vec{j} + \vec{k}$.

17. Consider the plane $x + 2y + z = 0$.
    (a) Find a normal vector $\vec{n}$ to this plane.
    (b) Let $\vec{u} = \vec{i} + \vec{j} + \vec{k}$. Write $\vec{u}$ as the sum of a vector $\vec{v}$ which is parallel to $\vec{n}$ and a vector $\vec{w}$ which is parallel to the plane.
    (c) Using part (b) find the (shortest) distance from the point $P = (1, 1, 1)$ to the plane. Hint: Note that the point $Q = (0, 0, 0)$ is on the plane and that $\vec{u} = \vec{QP}$.

18. Determine the distance from:
    (a) the point $(1, 3, -2)$ to the plane $2x + y - z = 1$.
    (b) the plane $2x + y - z = 1$ to the plane $2x + y - z = 6$. (Hint: Although there are many approaches, projections come in handy.)

19. True or false? Give a reason for your answer.
    (a) The vector $\vec{a} \times \vec{b}$ is parallel to the z-axis where $\vec{a}$ and $\vec{b}$ are shown in Figure 11 (both $\vec{a}$ and $\vec{b}$ are in the xy-plane).

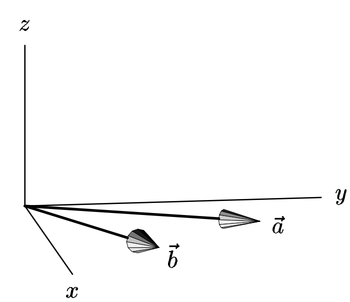


Figure 11

    (b) The vector $\vec{a} \times \vec{b}$ has a positive z component if $\vec{a}$ and $\vec{b}$ are as in part (a).

20. (a) Consider two vectors $\vec{v} = 2\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{w} = a\vec{i} + a\vec{j} - \vec{k}$. For what value(s) of a are $\vec{v}$ and $\vec{w}$ perpendicular?
    (b) For what values of a are $\vec{v}$ and $\vec{w}$ parallel?
    (c) Find the equation of the plane parallel to $2\vec{i} - \vec{j} + 3\vec{k}$ and to $3\vec{i} + 3\vec{j} - \vec{k}$ and containing the point $(1, 1, 1)$.

21. Given the points $P = (1, 2, 3)$, $Q = (3, 5, 7)$, and $R = (2, 5, 3)$, find:
    (a) The equation of the plane containing P, Q, and R.
    (b) The area of the triangle PQR.
    (c) The (perpendicular) distance from the point R to the line through P and Q. (Hint: Use Part(b))

22. The level curves of a function $z = f(x, y)$ are shown below. Assume that the scales along the x and y axes are the same. Arrange the following quantities in ascending order. Give a brief explanation for your reasoning.

$$f_x(P), \quad f_y(Q), \quad f_y(R), \quad f_x(S), \quad \text{the number } 0$$

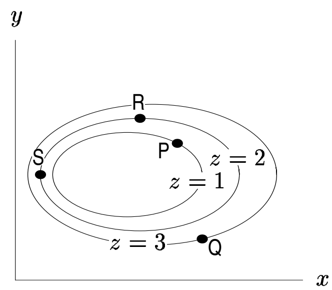


Figure 12

23. Suppose that the price P (in dollars) to purchase a used car is a function of C, its original cost (in dollars), and its age A (in years). So $P = f(C, A)$. What is the sign of $\dfrac{\partial P}{\partial C}$? Explain.

24. Find $\dfrac{\partial}{\partial x}(\ln(x^2y + 3))$

25. Find f_H if $f(H,T) = \dfrac{2H + T}{(5 - H)^3}$

26. The ideal gas law states that
$$PV = RT$$
for a fixed amount of gas, called a *mole* of gas, where P is the pressure (in atmospheres), V is the volume (in cubic meters), T is the temperature (in degrees Kelvin) and R is a positive constant.
   (a) Find $\partial P/\partial T$ and $\partial P/\partial V$.
   (b) A mole of a certain gas is at a temperature of $300°K$, a pressure of 1 atmosphere, and a volume of 0.02m^3. What is $\partial P/\partial V$ for this gas?
   (c) Explain the meaning of your answer to (b) in terms of temperature, pressure and volume.

27. Determine the tangent plane to $z = f(x, y) = 3e^{(x-y)} \ln x$ at $(x, y) = (1, 1)$.

28. A ball thrown from ground level (with initial speed v and at an angle α with the horizontal) hits the ground at a distance $s = \frac{v^2 \sin(2\alpha)}{g}$ ($g \approx 10 \frac{\text{m}}{\text{s}^2}$ is gravitational acceleration).
   (a) Find a reasonable domain for the values of α and v. (Hint: A football field is about $s = 100$ m long.)
   (b) Describe the shape of the graph, and/or sketch the graph of s as a function of (v, α).
   (c) Calculate the differential ds. What does the sign of $\frac{ds}{d\alpha}(20, \frac{\pi}{3})$ tell you?
   (d) Use the linearization of s about $(v, \alpha) = (20, \frac{\pi}{3})$ to estimate how α should change to get approximately the same distance s when v changes to 19 m/s.

29. The depth of a pond at the point with coordinates (x, y) is given by $h(x, y) = 2x^2 + 3y^2$ feet.
   (a) If a boat at the point $(-1, 2)$ is sailing in the direction of the vector $4\vec{i} + \vec{j}$, is the water getting deeper or shallower? At what rate? Assume that x and y are measured in feet.
   (b) In which direction should the boat at $(-1, 2)$ move for the depth to remain constant?

30. The directional derivative of $f(x, y)$ at $(1, 1)$ in the direction $\vec{i}$ is $\sqrt{2}$, and in the direction $(1/\sqrt{2})\vec{i} + (1/\sqrt{2})\vec{j}$ is -3. Find the directional derivative of f at $(1, 1)$ in the direction $\vec{u_3}$, where $\vec{u_3} = (2/\sqrt{13})\vec{i} + (3/\sqrt{13})\vec{j}$.

31. Write a paragraph explaining the meaning of and relationships between the partial derivative, the directional derivative and the gradient of a function f. Pay close attention to the meaning of these mathematical objects, as opposed to methods of computation. Do not simply list definitions: try to write as if you were explaining these objects to a friend who has had one year of one-variable calculus.

32. Find an equation for the tangent plane to the ellipsoid $(x - 1)^2 + 4(y - 2)^2 + (z - 3)^2 = 17$ at the point $(3, 3, 6)$.

33. The quantity z can be expressed as a function of x and y as follows: $z = f(x, y)$. Now x and y are themselves functions of r and θ, as follows: $x = g(r, \theta)$ and $y = h(r, \theta)$. Suppose you know that $g(1, \pi/2) = 0$, and

$h(1, \pi/2) = 1$. In addition, you are told that $f_x(0,1) = 2$, $f_y(0,1) = 3$, $g_r(1, \pi/2) = 5$, $g_\theta(1, \pi/2) = 7$, $h_r(1, \pi/2) = 9$, $h_\theta(1, \pi/2) = 11$. Find $\frac{\partial z}{\partial r}|_{(1,\pi/2)}$.

34. Let $w = 3x \cos y$.
    (a) If $x = u^2 + v^2$, $y = v/u$, find $\partial w/\partial u$ and $\partial w/\partial v$ at the point $(u, v) = (1, 1)$.
    (b) If $x = e^{-t}$ and $y = \ln t$, find $\partial w/\partial t$ at the point $t = 1$.

35. Find the following partial derivatives:
    (a) $H_P(1, 2)$ if $H(P, T) = \dfrac{3P}{2P + T}$,
    (b) f_{xy} if $f(x, y) = (xy)^7$.

36. (a) Find $\dfrac{\partial}{\partial P}\left(\dfrac{Pe^T}{T - e^P}\right)$. Do not simplify your answer.
    (b) Find the directional derivative of $f(x, y) = x^3y + xy^2$ at the point $(1, 2)$ in the direction of $3\vec{i} - 4\vec{j}$.
    (c) Find a unit vector perpendicular to both $\vec{i} - \vec{j} - \vec{k}$ and $2\vec{i} + \vec{k}$.
    (d) The ideal gas law states that $PV = nRT$ where n and R are constants. Find the rate of change of pressure with respect to volume in an experiment where the temperature remains constant.
    (e) Find the angle between the vector $\vec{c} = 2\vec{i} + 3\vec{j} + \vec{k}$ and the positive z-axis.

37. Consider the planes:
    I. $3x - 5y - z = 2$
    II. $5x = y + 3$
    III. $5x + 3y = 2$
    IV. $3x + 5y = 2$
    V. $3x + 5y + z = 2$
    VI. $y + 1 = 0$

    Without giving reasons, list all of the planes which:
    (a) Are parallel to the z-axis.
    (b) Are parallel to $3x = 5y + z + 7$.
    (c) Contain the point$(1, -1, 6)$.
    (d) Are normal to the cross product of the vectors $\vec{a} = 2\vec{i} + 3\vec{k}$ and $\vec{b} = 3\vec{i} - \vec{k}$.
    (e) Could be the tangent plane to a surface $z = f(x, y)$, where f is some function which has finite partial derivatives everywhere.

38. Let R be the region in the first quadrant bounded by the x and y-axes and the line $x + 2y = 6$. Write $\int_R \sqrt{x + 2y}\, dA$ as an iterated integral in two different ways and hence evaluate it. Your answer should contain a sketch of the region R.

39. Consider the integral $\int_1^5 \int_{2x}^{10} f(x, y) dy dx$
    (a) Sketch and label clearly the region over which the integration is being performed.
    (b) Rewrite the integral with the integration performed in the opposite order.

40. Find the volume under the graph of $f(x, y) = xe^y$ lying over the triangle with vertices $(0, 0)$, $(2, 2)$, $(4, 0)$.

41. Set up *but do not evaluate* a (multiple) integral that gives the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

42. Calculate the following integrals exactly. (Your answer should not be a decimal approximating the true answer, but should be exactly equal to the true answer. Your answer may contain e, π, $\sqrt{2}$, and so on.)
    (a) $\displaystyle\int_0^1 \int_0^z \int_0^y x^2y^3z^4\, dx\, dy\, dz$
    (b) $\displaystyle\int_0^6 \int_0^3 (\cos 3y) \sin(2x + 5)\, dx\, dy$
    (c) $\displaystyle\int_3^4 \int_0^y y^2e^{xy}\, dx\, dy$

43. Convert the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dy\,dx$ to polar coordinates and hence evaluate it exactly. Sketch the region R over which the integration is being performed.

44. Calculate the following integrals:

(a) $\int_0^2 \int_0^y x^5 y^7\,dx\,dy$,

(b) $\int_R r\cos\theta\,dA$ where R is the shaded region shown in Figure 13.

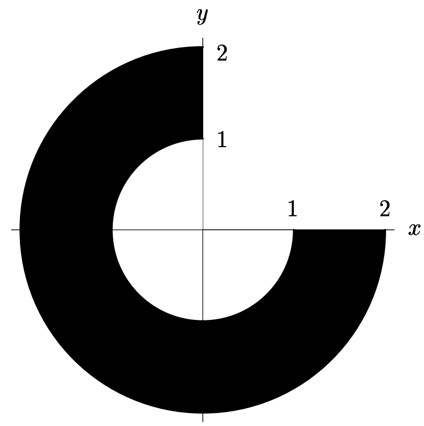


Figure 13

45. A city by the ocean surrounds a bay, and has the semiannular shape shown in Figure 14.

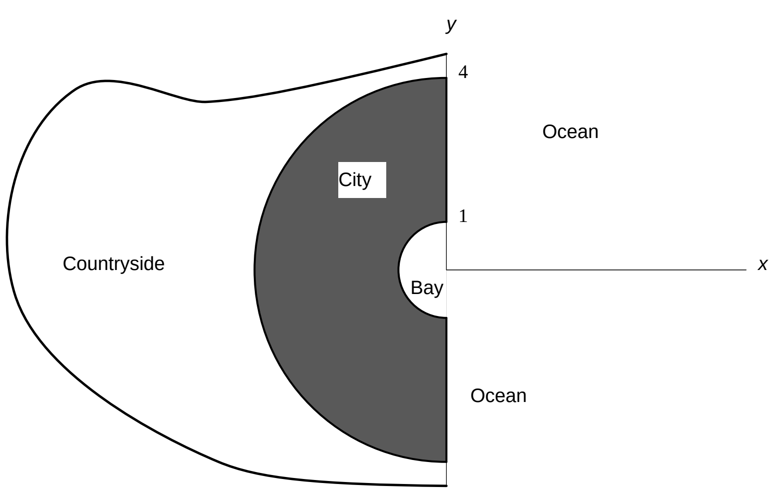


Figure 14

(a) The population density of the city (in thousands of people per square mile) is given by the function $\delta(r,\theta)$, where r and θ are the usual polar coordinates with respect the x and y-axes indicated in Figure 14, and the distances indicated on the y-axis are in miles. Set up a double integral in polar coordinates that would give the total population of the city.

(b) The population density decreases as you move away from the shoreline of the bay, and also decreases the further you have to drive to get to the ocean. Which of the following functions best describes this situation?

(a) $\delta(r,\theta) = (4-r)(2+\cos\theta)$

(b) $\delta(r, \theta) = (4 - r)(2 + \sin\theta)$
(c) $\delta(r, \theta) = (r + 4)(2 + \cos\theta)$

(c) Evaluate the integral you set up in (a) with the function you chose in (b), and give the resulting estimate for the population.

46. Evaluate exactly the integral $\int_R y\, dA$, where R is each of the regions shown in Figure 15 and Figure 16.

(a)

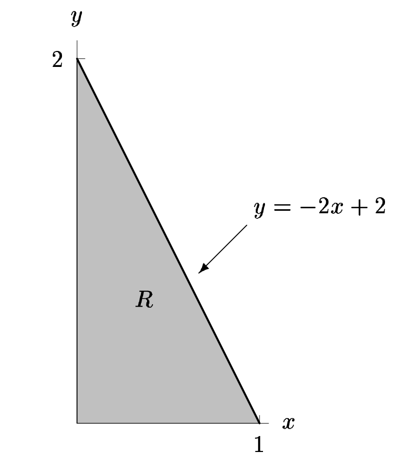


Figure 15

(b)

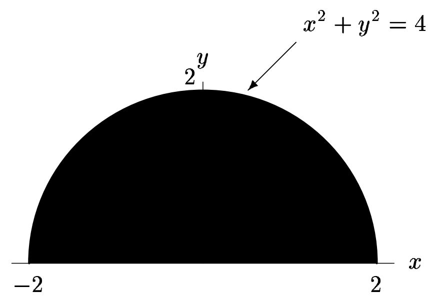


Figure 16

47. Consider the volume between a cone centered along the positive z-axis, with vertex at the origin and containing the point $(0, 1, 1)$, and a sphere of radius 3 centered at the origin.

(a) Write a triple integral which represents this volume. Use spherical coordinates.
(b) Evaluate the integral.

48. A cylindrical tube of radius 2 cm and length 3 cm contains a gas. Since the tube is spinning around its axis, the density of the gas increases with its distance from the axis. The density, D, at a distance of r cm from the axis is $D(r) = 1 + r$ gm/cc.

(a) Write a triple integral representing the total mass of the gas in the tube.
(b) Evaluate this integral.

49. Evaluate $\int_W z\, dV$ where W is the first octant portion of the ball of radius 2 centered at the origin. Do this in two ways, using spherical and cylindrical coordinates.

50. Give parameterizations for the following:

(a) A circle of radius 3 in the plane, centered at origin, traversed clockwise.
(b) A circle of radius 2 in 3-space perpendicular to the y-axis.

51. Consider the plane $2x + y - 5z = 7$ and the line with parametric equation $\vec{r} = \vec{r}_0 + t\vec{u}$.

(a) Give a value of $\vec{u}$ which makes the line perpendicular to the plane.

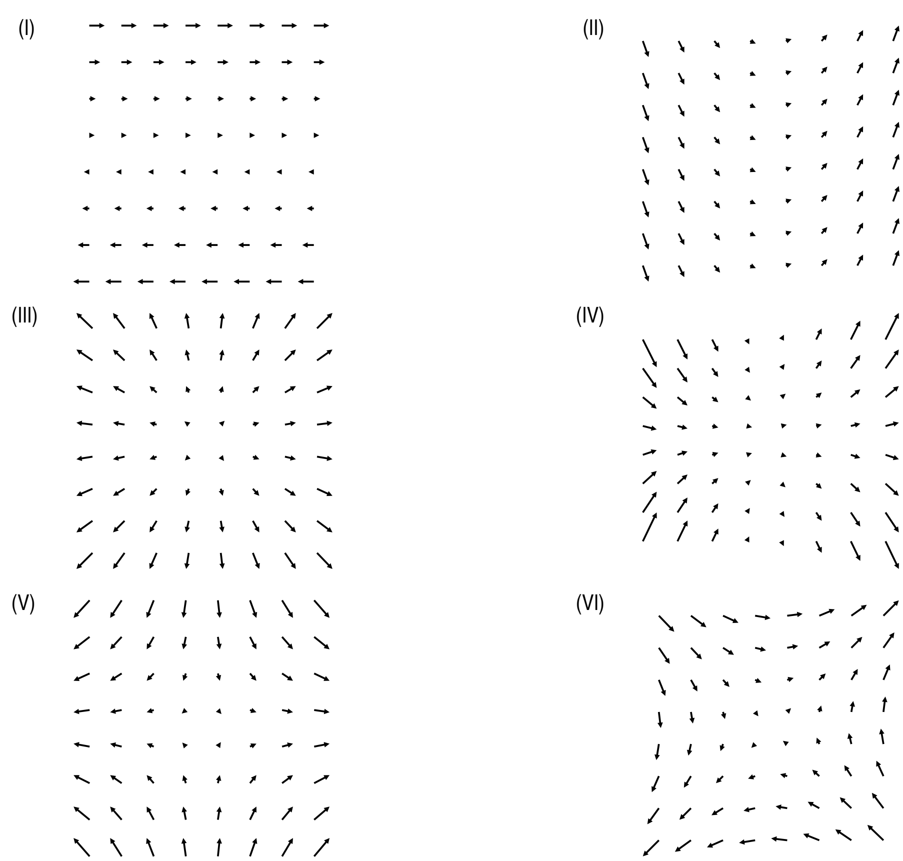
(b) Give a value of $\vec{u}$ which makes the line parallel to the plane.
(c) Give values for $\vec{r_0}$ and $\vec{u}$ which make the line lie in the plane.

52. The equation $\vec{r} = \vec{i} + 5\vec{j} + 3\vec{k} + t(\vec{i} - \vec{j} + \vec{k})$ parameterizes a line through the points $(1,5,3)$, $(2,4,4)$, and $(0,6,2)$. What value of t gives each of these three points?

53. A child is sliding down a helical slide. Her position at time t after the start is given in feet by $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + (10-t)\vec{k}$. The ground is the xy-plane.
   (a) When is the child 6 feet from the ground?
   (b) At time $t = 2\pi$, the child leaves the slide on the tangent to the slide at that point. What is the equation of the tangent line?

54. The equation $\vec{r} = x\vec{i} + y\vec{j} = 2\cos(2\pi t/360)\vec{i} + 2\sin(2\pi t/360)\vec{j}$ describes the motion of a particle moving on a circle. Assume x and y are in miles and t is in days.
   (a) What is the radius of the circle?
   (b) What is the period of the circular motion?
   (c) What are the velocity and the speed of the particle when it passes through the point $(0,2)$?

55. For each of the following vector fields, identify which one of the following figures could represent it. The scales in the x and y directions are the same. No reasons need be given. (a) $x\vec{i} + y\vec{j}$ (b) $x\vec{i} - y\vec{j}$ (c) $y\vec{i} + x\vec{j}$ (d) $y\vec{i}$ (e) $\vec{i} + x\vec{j}$ (f) $x^2\vec{i} + xy\vec{j}$



56. The following equations (i)-(viii) represent curves or surfaces. Match each of the equations with one of the geometric descriptions (a)-(g). Write your answers in the spaces to the right of the equations. If none of (a)-(g) applies, write "None." (Note: ρ, ϕ, θ are spherical coordinates; r, θ, z are cylindrical coordinates.)
   (i) $r = 2,\ 0 \le z \le 10$______________

(ii) $x = t, y = 2t, z = 3t, 0 \le t \le 10$______________

(iii) $x = t^2, y = 2t^2, z = 3t^2, 0 \le t \le 10$______________

(iv) $z = 10, 0 \le r \le 10$______________

(v) $\rho = \frac{1}{\cos\phi}, 0 \le \phi \le \frac{\pi}{3}$______________

(vi) $\rho = \cos\phi, \theta = \frac{\pi}{4}, 0 \le \phi \le \frac{\pi}{3}$______________

(vii) $\phi = \frac{\pi}{4}, 0 \le \rho \le 10$______________

(viii) $z = r^2, \theta = \frac{\pi}{4}, 0 \le r \le 10$______________

(a) Part of a line through the origin.
(b) Part of a parabola through the origin.
(c) Line or curve through $(x, y, z) = (0, 0, 1)$ in a horizontal plane.
(d) Line or curve through $(x, y, z) = (0, 0, 1)$ in a vertical plane.
(e) Disc.
(f) Part of a cone.
(g) Part of a cylinder.

57. Explain clearly what is meant by a line integral $\int_C \vec{F} \cdot d\vec{r}$.

58. (a) Given the graph of the vector field, $\vec{F}$, shown in Figure 17, give a possible formula for such a field.

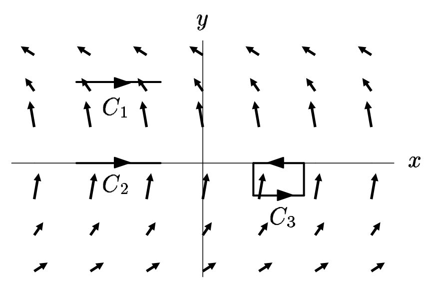


Figure 17

(b) List the following quantities in increasing order:
(i) $\int_{C_1} \vec{F} \cdot d\vec{r}$,
(ii) $\int_{C_2} \vec{F} \cdot d\vec{r}$,
(iii) $\int_{C_3} \vec{F} \cdot d\vec{r}$.

59. Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x + z)\vec{i} + z\vec{j} + y\vec{k}$ and C is the line from the point $(2, 4, 4)$ to the point $(1, 5, 2)$.

60. If $\vec{F} = x^2\vec{i} + z\sin(yz)\vec{j} + y\sin(yz)\vec{k}$, compute $\int_c \vec{F} \cdot d\vec{r}$ where C is the curve from $A(0, 0, 1)$ to $B(3, 1, 2)$ shown in Figure 18 (messy computation can be avoided).

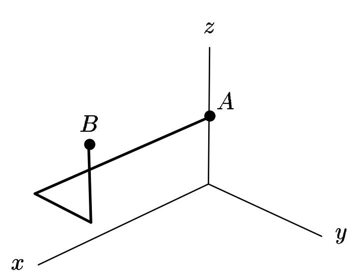


Figure 18

61. Which of the following vector fields is a gradient vector field? For any that is, find the potential function f such that $\vec{F} = \text{grad}\, f$.

(a) $\vec{F} = x\vec{i} + y\vec{j}$
(b) $\vec{F} = y\vec{i} - x\vec{j}$

62. Let $\vec{F}(x, y)$ be the vector field $\vec{F}(x, y) = (x^2 + x)\vec{j}$. Let C_1 be the line from $(0, 0)$ to $(1, 1)$. Let C_2 be the unit circle traveled counterclockwise.

(a) Give a graph of this vector field.
(b) Give a parameterization of C_1 and C_2.
(c) Find $\int_{C_1} \vec{F} \cdot d\vec{r}$.
(d) Find $\int_{C_2} \vec{F} \cdot d\vec{r}$ using Green's theorem.

63.

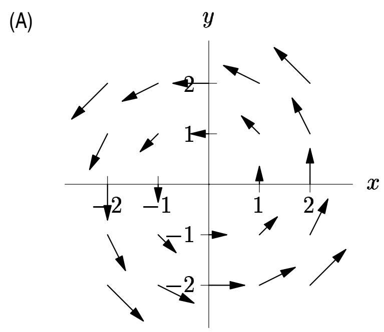


Figure 19:

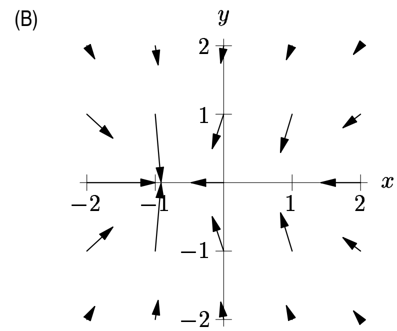


Figure 20:

(a) Which of the two vector fields shown is clearly not conservative? Explain why.
(b) In the vector field you chose in (a) sketch a closed curve around which the circulation is nonzero.

64. Consider four circular disks, S_1, S_2, S_3, S_4 each of radius 2 and centered on an axis, where:

- S_1 is perpendicular to the x-axis at $x = 5$ with normal in the direction of increasing x.
- S_2 is perpendicular to the x-axis at $x = 8$ with normal in the direction of decreasing x.
- S_3 is perpendicular to the y-axis at $y = -6$ with normal in the direction of decreasing y.
- S_4 is perpendicular to the z-axis at $z = 7$ with normal in the direction of increasing z.

Consider the vector field $\vec{F} = x\vec{i} + y\vec{j} + (z + x)\vec{k}$.

(a) Which of the flux integrals
  (i) $\int_{S_1} \vec{F} \cdot d\vec{A}$
  (ii) $\int_{S_2} \vec{F} \cdot d\vec{A}$
  (iii) $\int_{S_3} \vec{F} \cdot d\vec{A}$
  (iv) $\int_{S_4} \vec{F} \cdot d\vec{A}$
  are positive? negative? zero? How do you know?
(b) Find $\int_{S_1} \vec{F} \cdot d\vec{A}$.

65. What is the flux of the vector field $\vec{F} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ through a circle in the xy-plane of radius 3 oriented upward with center at the origin.

66. Let $\vec{H} = 2x\vec{i} - 3xy\vec{j} + xz^2\vec{k}$,

(a) What is div $\vec{H}(x, y, z)$?
(b) What is $\int_S \vec{H} \cdot d\vec{A}$, where S is the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, and $(1, 1, 1)$?

67. Questions (a)-(c) refer to the two-dimensional fluid flow

$$\vec{v}(x, y) = (x^2 + y^2)^a x\vec{i} + (x^2 + y^2)^a y\vec{j},$$

where a is a constant. (We allow a to be negative, so $\vec{F}$ may or may not be defined at $(0,0)$.)

(a) Is the fluid flowing away from the origin, toward it, or neither?
(b) Calculate the divergence of $\vec{F}$. Simplify your answer.
(c) For what values of a is div $\vec{F}$ positive? Zero? Negative?
(d) What does your answer to (c) mean in terms of flow? How does this fit in with your answer to (a)?

68. (a) If $\vec{F}$ is a vector field, what is meant by div $\vec{F}$ at a point P? Your answer should say whether div $\vec{F}$ is a vector or a scalar, and explain what div $\vec{F}$ tells you if $\vec{F}$ represents the velocity of a moving fluid.
(b) How is div $\vec{F}$ defined?
(c) If $\vec{F} = ye^{x^2}\vec{i} + xye^y\vec{j} + z\cos(xy)\vec{k}$, find div $\vec{F}$.

69. An oceanographic vessel suspends a paraboloid-shaped net below the ocean at depth of 1000 feet, held open at the top by a circular metal ring of radius 20 feet, with bottom 100 feet below the ring and just touching the ocean floor. Set up coordinates with the origin at the point where the net touches the ocean floor and with z measured upward. Water is flowing with velocity:

$$\vec{F} = 2xz\vec{i} - (1100 + xe^{-x^2})\vec{j} + z(1100 - z)\vec{k}.$$

(a) Write down an iterated integral I_1 for the flux of water through the net (oriented from inside to outside). Include the limits of integration but do not evaluate.
(b) Use the Divergence Theorem to compare this integral with the flux I_2 across the circular disk which is the open top of the paraboloid-shaped net. In this way evaluate I_1.

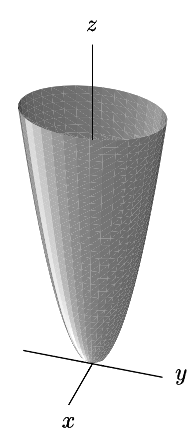


Figure 21

:w

70. (a) What is meant by curl $\vec{F}$ where $\vec{F}$ is a vector field? Is curl $\vec{F}$ a vector or a scalar?
(b) How is curl $\vec{F}$ defined?
(c) Find curl$(-y\vec{i} + x\vec{j} + (z^2 + y^2 + x^2)\vec{k})$.

71. The vector fields in Figures 22–24 each show a vector field of the form $\vec{F} = F_1\vec{i} + F_2\vec{j}$. Assume F_1 and F_2 depend only on x and y and not on z. For each vector field, circle the best answers. No reasons need be given.

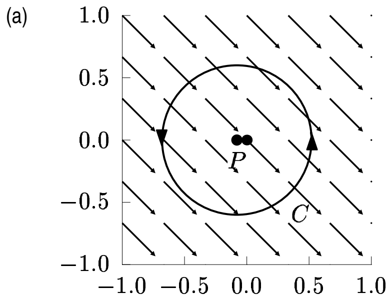


Figure 22:

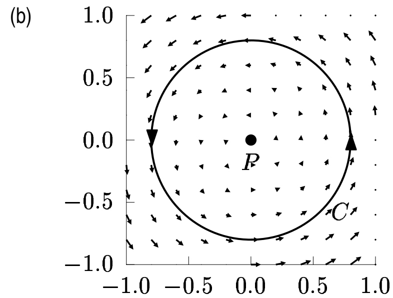


Figure 23:

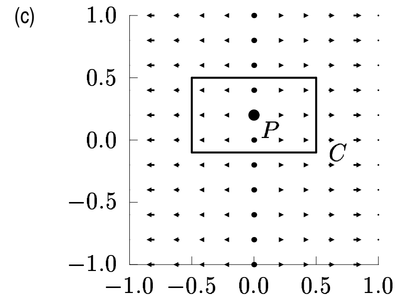


Figure 24:

(i) $\int_C \vec{F} \cdot d\vec{r}$ is
positive negative zero

(ii) div $\vec{F}$ at P is
positive negative zero

(iii) curl $\vec{F}$ at P has a
positive z component negative z component zero z component

(iv) $\vec{F}$
could be a gradient field could not be a gradient field

72. If curl $\vec{F}$ is parallel to the x-axis for all x, y, and z and if C is a circle in the xy-plane, then the circulation of $\vec{F}$ around C is zero. True or false?

73. Suppose W is the object consisting of two solid cylinders meeting at right angles at the origin. One cylinder is centered on the y-axis between $y = -5$ and $y = 5$ with radius 2 and the other is centered on the x-axis between $x = -5$ and $x = 5$ with radius 2. Let S be the whole surface of W except for the circular end of the cylinder centered at $(0, 5, 0)$. The boundary of S is a circle, C; the surface S is oriented outward. Let $\vec{F} = (3x^2 + 3z^2)\vec{j}$. You are also told that $\vec{F} = \text{curl}(z^3\vec{i} + y^3\vec{j} - x^3\vec{k})$.

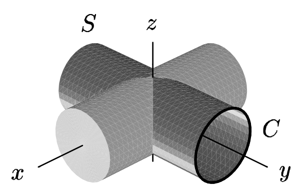


Figure 25

(a) Suppose you want to calculate $\int_S \vec{F} \cdot d\vec{A}$. Write down two other integrals which have the same value as $\int_S \vec{F} \cdot d\vec{A}$. Justify your answer.
(b) Find $\int_S \vec{F} \cdot d\vec{A}$ whatever by method is easiest.

74. Mark the following statements T (true) or F (false). No reasons need be given.
$\vec{v}, \vec{w}$ are arbitrary vectors in 3-space
$f(x, y)$ is an arbitrary differentiable function.
$\vec{F}, \vec{G}$ are arbitrary differentiable vector fields in 3-space.
____________ $\vec{v} \times \vec{w}$ is always parallel to $\vec{v}$.
____________ $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.
____________ div $\vec{G}$ is a vector field.
____________ grad f is a vector field.
____________ $f_{\vec{u}}(a, b) \geq -||\,\text{grad}\, f(a, b)||$ for all unit vectors $\vec{u}$ in 2-space.
____________ At each point, curl $\vec{F}$ is always perpendicular to $\vec{F}$.
____________ $2x + 3y + 5z = 1$ represents a line in 3-space.
____________ The point $(81, -9, 3)$ lies on the curve $(t^2, -t, \sqrt{t})$.

75. Mark each of the following quanties as V (vector), S (Scalar), or N (not defined). No reasons need be given. You are told that:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
$\vec{u}$ is any unit vector in 3-space
$f(x, y, z)$ is any differentiable function
$\vec{F}$ is any differentiable vector field.

____________ $(\vec{u} \times \vec{r}) \times \vec{r}$	____________ $f_{\vec{u}}(1,2,3)$
____________ $(\text{grad}\, f) \cdot \vec{r}$	____________ $f_{\vec{u}}(1,2,3) \times \vec{r}$
____________ $\text{div}\, f$	____________ $(\text{div}\, \vec{r})\vec{r}$
____________ $(\text{curl}\, \vec{F}) \times \vec{r}$	____________ $\text{grad}\, \vec{F}$
____________ $\text{div}(\vec{F} \times \vec{r})$	____________ $\text{div}(f\vec{r})$

76. Let

$$\vec{F} = 2xy^2\vec{i} + (2yx^2 + 2y)\vec{j}$$

and

$$\vec{G} = (y + x)\vec{i} + (y - x)\vec{j}\,.$$

Let C_1 be the curve composed of five stright line segments connecting the origin to $(1, 2)$ to $(1, -1)$ to $(-1, -1)$ to $(-1, 1)$ back to the origin. Let C_2 be the curve composed of three-quarters of a circle of radius 2 from $(2, 0)$ counterclockwise to $(0, -2)$. Calculate the following four integrals:

i. The line integral of $\vec{F}$ over C_1
ii. The line integral of $\vec{G}$ over C_1
iii. The line integral of $\vec{F}$ over C_2
iv. The line integral of $\vec{G}$ over C_2