4 Measurable functions

Let E be a measurable set and $f_n : E \to \overline{\mathbb{R}}$ a sequence of measurable functions.

(A) $f_n \to f$ pointwise in E if $f_n(x) \to f(x)$ for all $x \in E$.

(B) $f_n \to f$ pointwise almost everywhere in E if $f_n(x) \to f(x)$ for almost all $x \in E$.

(C) $f_n \to f$ almost uniformly in E if for each $\epsilon > 0$ there exists a measurable set $F \subset E$ such that $|E \setminus F| \leq \epsilon$ and $f_n \to f$ uniformly in F.

(D) $f_n \to f$ in measure in E if for each $\epsilon > 0$, $\lim_{n\to\infty} |\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}| = 0$.

(E) f_n is a Cauchy sequence in measure in E if for each $\epsilon, \delta > 0$, there exists N such that $|\{x \in E : |f_n(x) - f_m(x)| \geq \epsilon\}| \leq \delta$ for all $n, m \geq N$.

$f : E \to \mathbb{R}$ is a measurable function if $f^{-1}((-\infty, a))$ is a Lebesgue measurable set for all $a \in \mathbb{R}$.

$f : E \to \mathbb{R}$ is a Borel measurable function if $f^{-1}((-\infty, a))$ is a Borel set for all $a \in \mathbb{R}$.

Problem 4.1 If $f_n \to f$ almost uniformly in E, then $f_n \to f$ in measure in E.

Problem 4.2 If $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ a.e.

Problem 4.3 If f_n is a Cauchy sequence in measure, then there exists a subsequence f_{n_k} that is a Cauchy sequence almost uniformly.

Problem 4.4 If f_n is a Cauchy sequence in measure, then there exists f measurable such that $f_n \to f$ in measure.

Problem 4.5 Give an example of a sequence f_n that converges in measure but does not converge a.e.

Problem 4.6 If $|E| < \infty$ and $f_n \to f$ a.e. in E, then $f_n \to f$ in measure. Show that this is false if $|E| = \infty$.

Problem 4.7 If $f_n \to f$ in measure, then there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Problem 4.8 If $f_n \to f$ in measure, then f_n is a Cauchy sequence in measure.

Problem 4.9 Let $f, f_k : E \to \mathbb{R}$ be measurable functions. Prove that

1. If $f_k \to f$ in measure, then any subsequence f_{k_j} contains a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$, .
2. Suppose $|E| < \infty$. If any subsequence f_{k_j} contains a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$, then $f_k \to f$ in measure.

Hint: for (2) suppose by contradiction that $f_k \not\to f$ in measure. Then there exists $\epsilon_0 > 0$ such that $|\{x \in E : |f_k(x) - f(x)| \geq \epsilon_0\}| \not\to 0$. Hence there is an increasing sequence $k_j \to \infty$ such that $|\{x \in E : |f_{k_j}(x) - f(x)| \geq \epsilon_0\}| \geq r > 0$, for some $r > 0$ and for all j. By hypothesis there is a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$. Now use Egorov to get a contradiction.

Problem 4.10 Let $f : E \to \mathbb{R}$ be a measurable function. Prove that if $B \subset \mathbb{R}$ is a Borel set then $f^{-1}(B)$ is measurable.

Hint: consider $\mathcal{A} = \{A \subset \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$ and show that $\mathcal{A}$ is a σ-algebra that contains the open sets of $\mathbb{R}$.

Problem 4.11 If $f : E \to \mathbb{R}$ is a measurable function and $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $g \circ f$ is measurable.