Problem1 .
Show that for every real number k,

$$g(x) = (x + k\sqrt{x})^2$$

satisfies the differential equation

$$y' - \frac{y}{x} = y^{1/2}.$$

Let $y = g(x)$

$$y' = 2\left(1 + \frac{k}{2\sqrt{x}}\right)(x + k\sqrt{x})$$

$$= \left(2 + \frac{k}{\sqrt{x}}\right)(x + k\sqrt{x}) \leftarrow \text{1pt}$$

$$y' - \frac{y}{x} = \left(2 + \frac{k}{\sqrt{x}}\right)(x + k\sqrt{x}) - \frac{(x + k\sqrt{x})^2}{x}$$

1pt

$$= (x + k\sqrt{x})\left(2 + \frac{k}{\sqrt{x}} - \frac{x + k\sqrt{x}}{x}\right)$$

$$= (x + k\sqrt{x})\left(1 + \frac{k\sqrt{x}}{x} - 1 - \frac{k\sqrt{x}}{x}\right)$$

$$= x + k\sqrt{x} \leftarrow \text{0.5pt}$$

$$= y^{\frac{1}{2}} \leftarrow \text{0.5pt}$$

Problem2 .

Use the reduction of order method to solve the equation

$$y'' - e^y y' = 0$$

1pt

Let $u = y'$ 1pt $\Rightarrow$ $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = u \cdot \frac{du}{dy}$

The equation becomes

$$u\frac{du}{dy} - e^y u = 0 \leftarrow \text{1pt}$$

If $u = 0$, we obtain the trivial solutions $y = C$.

Assume $u \neq 0$. We have 1pt

$$\frac{du}{dy} = e^y \iff du = e^y dy \Rightarrow \int du = \int e^y dy + C_1$$

$$\iff u = e^y + C_1 \iff \frac{dy}{dx} = e^y + C_1 \iff \frac{dy}{e^y + C_1} = dx$$

1pt

$$\Rightarrow \int \frac{dy}{e^y + C_1} = \int dx + C_2 \iff \leftarrow \text{1pt}$$

$$\frac{1}{C_1}\left(y - \ln|e^y + C_1|\right) = x + C_2 \iff \leftarrow \text{1pt}$$

$$y - \ln|e^y + C_1| = C_1 x + C_1 C_2 \iff$$

$$y - C_1 x - C_3 = \ln|e^y + C_1| \iff$$

$$|e^y + C_1| = e^{-C_3} \cdot e^{y - C_1 x} \Rightarrow$$

$$e^y + C_1 = \pm e^{-C_3} \cdot e^{y - C_1 x} \iff$$

$$e^y = A e^{y - Cx} + C$$ where $C = -C_1$ and $A = \pm e^{-C_3}$

1pt

Problem3 . Consider the differential equation

$$3(1+\frac{dy}{dx})+4(x+y) = 2x(x+y)^{3/2}. \qquad (I)$$

(a) Use an appropriate substitution u to write the differential equation (I) in the form

$$\frac{du}{dx}+\frac{4}{3}u = \frac{2}{3}u^{3/2}x.$$

1pt 1pt

Let $u = x+y$, $\frac{du}{dx} = 1+\frac{dy}{dx}$. (I) becomes

$$3\frac{du}{dx}+4u = 2xu^{3/2} \iff$$

$$\frac{du}{dx}+\frac{4}{3}u = \frac{2}{3}xu^{3/2} \leftarrow \text{1pt}$$

(b) Solve the differential equation in part (a).

This is a Bernoulli equation with $P(x) = \frac{4}{3}$, $Q(x) = \frac{2}{3}x$, and $n = \frac{3}{2}$. 1pt

Doing the substitution $v = u^{1-\frac{3}{2}} = u^{-\frac{1}{2}}$, we get

$$\frac{dv}{dx}-\frac{2}{3}v = -\frac{x}{3}$$

1pt This is a linear first order equation. By the method of integrating factor, we have

$$\frac{d}{dx}\left[ve^{-\frac{2}{3}x}\right] = -\frac{x}{3}e^{-(2/3)x} \Rightarrow$$

$$v\cdot e^{-\frac{2x}{3}} = -\frac{1}{3}\int xe^{-\frac{2}{3}x}\,dx + C$$

1pt

Integrating by parts and dividing by $e^{-\frac{2x}{3}}$ we obtain

$$v = \frac{1}{2}x+\frac{3}{4}+Ce^{+\frac{2}{3}x} \leftarrow \text{1pt} \Rightarrow$$

$$u = \frac{1}{\left[\frac{1}{2}(x+\frac{3}{2})+Ce^{+(2/3)x}\right]^2} \leftarrow \text{1pt}$$

Problem4 . Determine the value of m for which the differential equation is exact and then solve it.

$$\frac{dy}{dx} = \frac{x+my}{(m-1)x+y} \iff$$

$$(x+my)\,dx - [(m-1)x+y]\,dy = 0 \leftarrow \text{1pt}$$

Let $M(x,y) = x+my$, and $N(x,y) = -[(m-1)x+y]$

Both M and N are continuous with continuous partial derivatives on the xy-plane.

By the exactness criterion, the equation is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff m = 1-m$

$\iff m = \frac{1}{2}$. When $m = \frac{1}{2}$, the equation

2pts

becomes

$$\left(x+\frac{1}{2}y\right)dx + \left(\frac{1}{2}x - y\right)dy = 0$$

1pt

$$\frac{\partial F}{\partial x} = x + \frac{1}{2}y \Rightarrow F(x,y) = \frac{x^2}{2} + \frac{xy}{2} + g(y)$$

1pt

$$\frac{\partial F}{\partial y} = \frac{1}{2}x - y \Rightarrow \frac{x}{2} + g'(y) = \frac{x}{2} - y \Rightarrow$$

1pt

$$g'(y) = -y \quad \text{and} \quad g(y) = -\frac{y^2}{2} + c$$

1pt

$$F(x,y) = \frac{x^2}{2} + \frac{xy}{2} - \frac{y^2}{2}$$

The general solution of the equation is given by

$$\frac{x^2}{2} + \frac{xy}{2} - \frac{y^2}{2} = c \iff x^2 + xy - y^2 = c$$

1pt

Problem5. Solve the initial value problem

$$x\frac{dy}{dx} - y = \frac{x^3}{y}e^{y/x}, \qquad y(1) = 0$$

$$\frac{dy}{dx} = \frac{y}{x} + x\left(\frac{x}{y}\right)e^{y/x} \longleftarrow \text{1pt}$$

1pt

If we let $u = \frac{y}{x} \Leftrightarrow y = xu \Rightarrow$

1pt

$$\frac{dy}{dx} = u + x\frac{du}{dx} \Leftrightarrow$$

$$u + \frac{x}{u}e^{u} = u + x\frac{du}{dx} \Leftrightarrow$$

$$x\frac{du}{dx} = \frac{x}{u}e^{u}$$ (Separable) and

1pt

$$u e^{-u}\,du = dx$$

Integrating by parts, we obtain

1pt

$$-ue^{-u} - e^{-u} = x + C \Leftrightarrow$$

$$u + 1 = (C_1 - x)e^{u} \quad (C_1 = -C)$$

$$\Leftrightarrow \frac{y}{x} + 1 = (C_1 - x)e^{y/x} \Leftrightarrow$$

1pt

$$y + x = x(C_1 - x)e^{y/x} \qquad y(1) = 0$$

$$\Rightarrow 1 = C_1 - 1 \Leftrightarrow C_1 = 2 \quad \text{1pt.}$$

Problem 6.
(a) Find an explicit expression for the general solution of the differential equation

$$x\frac{dy}{dx} = y^2 - 25.$$

Separating variables, we obtain

1pt

$$\frac{dy}{y^2-25} = \frac{dx}{x} \Rightarrow \int \frac{dy}{(y-5)(y+5)} = \int \frac{dx}{x} + C$$

By partial fraction decomposition we get

$$\frac{1}{10}\int \left(\frac{1}{y-5} - \frac{1}{y+5}\right) dy = \ln|x| + C \Rightarrow$$

1pt $(C_1 = 10C)$

$$\ln\left|\frac{y-5}{y+5}\right| = \ln x^{10} + C_1$$

$$\Rightarrow \left|\frac{y-5}{y+5}\right| = e^{C_1} x^{10} \Rightarrow \frac{y-5}{y+5} = A x^{10} \quad (A = \pm e^{C_1})$$

1pt

$$\Rightarrow \quad y = \frac{5(1 + A x^{10})}{1 - A x^{10}}$$

1pt

(b) Find if exist the **singular** solution (s).

1pt

$y = \pm 5$ are two obvious solutions

$y = 5$ can be recovered from the general solution by selecting $A = 0$

1pt

1pt

$y = -5$ cannot be recovered by selecting A

or $y = -5$ is the only singular solution

Problem7 . It is of considerable interest to policy makers to model the spread of information through a population.

Assume the population is of a constant size M.

If the information is spread by word of mouth, the rate of spread of information is believed to be proportional to the product of the number of people who know and the number who don't. Write a differential equation for the number of people having the information by time t.

Let P be the number of people having the information at time t. We have

2 pts $$\frac{dP}{dt} = kP(M-P), \quad k>0$$

Problem8 . Newton's law of cooling states that a body of uniform composition when placed in an environment with a constant temperature will approach the temperature of the environment at a rate that is proportional to the difference in temperature of the body and the environment.

A turkey has been refrigerated for several days and has a uniform temperature of $40°F$. An oven is preheated to $325°F$. The turkey is placed in the oven for 20 minutes and then taken out and its temperature is found to be $60°F$.

How long does the turkey have to stay in the oven to have a temperature of $185°F$?

Let T be the temperature of the turkey at time t. By Newton's Law

2 pts $\frac{dT}{dt} = k(325-T)$. Solving by separating variables, we obtain

1 pt $T = Ce^{-kt} + 325$. Using $T(0)=40$,

we get $C = -285$. Using $T(20) = 60$, 1 pt

we get $k = \frac{\ln(57/53)}{20}$ 1 pt

$$T = -285\, e^{-\frac{\ln(57/53)}{20}t} + 325$$

Now we need to find t so that $T(t) = 185$ 1 pt

We solve for t to obtain

$$t = -20\,\frac{\ln(28/57)}{\ln(57/53)}$$ 1 pt