

Witten Laplacian Methods for The Decay of Correlations

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Abstract

The aim of this paper is to apply direct methods to the study of integrals of the type that appear naturally in Statistical Mechanics and Euclidean Field Theory. We provide weighted estimates leading to the exponential decay of the two point correlation functions for certain classical unbounded models. The method involves the study of the solution of the Witten Laplacian equation associated with the Hamiltonian of the system.

1 Introduction

In these notes, we study partial differential equation techniques for problems coming from equilibrium Statistical Mechanics and Euclidean Field theory. In the context of classical equilibrium Statistical Mechanics, one is interested in a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components. Consider, for example a piece of ferromagnetic metal (like iron, cobalt, or nickel) in thermal equilibrium. The piece consists of a very large number of atoms which are located at the sites of a crystal lattice Λ . Each atom shows a magnetic moment which can be visualized as a vector in \mathbb{R}^3 . This magnetic moment is called the *spin* of the atom and represents the orientation of the atom in the lattice. The set S of all possible orientations of the spins, is called the *state space* of the system. Each element i of Λ is called a (lattice) *site*. A particular configuration of the total system will be described by an element $x = (x_i)_{i \in \Lambda}$ of the product space $\Omega = S^\Lambda$. This set Ω is called the configuration space.

The physical system considered above is characterized by a sharp contrast: the microscopic structure is enormously complex, and any measurement of microscopic quantities is subject to Statistical fluctuations. The macroscopic behavior, however, can be described by means of a few parameters such as magnetization and temperature, and macroscopic measurement leads to apparently deterministic results. This contrast between the microscopic and the macroscopic level is the starting point of Classical Statistical Mechanics as developed

by Maxwell, Boltzman, and Gibbs. Their basic idea may be summarized as follows: The microscopic complexity may be overcome by a statistical approach, and the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers. According to this philosophy, it is not adequate to describe the state of the system by a particular element x of the configuration space Ω . The system's state should rather be described by a family of S -valued random variables or (if we pass to the joint distribution of these random variables), by a probability measure μ on Ω consistent with the available partial knowledge of the system. In particular, μ should take account of the a priori assumption that the system is in thermal equilibrium.

Which kind of probability measure on Ω is suitable to describe a physical system in equilibrium? The term equilibrium clearly refers to the notion of forces and energies that act on the system. Thus one needs to define a Hamiltonian Φ which assigns to each configuration x a potential energy $\Phi(x)$. In the physical system above, the essential contribution to the potential energy comes from the interaction of the microscopic components of the system and a possible external force. As soon as a Hamiltonian Φ have been specified, the answer to the question is generally believed to be the probability measure

$$d\mu(x) = Z^{-1} e^{-\beta\Phi(x)} d\lambda(x).$$

Here $d\lambda$ refers to a suitable a priori measure (for example the counting measure if Ω is finite), β is a positive number which is proportional to the inverse of the absolute temperature and $Z > 0$ is a normalization constant. The above measure μ is called *the Boltzmann-Gibbs distribution*.

As we have mentioned above the number of atoms in a ferromagnet is extremely large. Consequently, the set Λ in our mathematical model should be very large. According to a standard rule of a mathematical thinking, the intrinsic properties of large objects can be made manifest by performing suitable limiting procedures. It is therefore a common practice in Statistical Physics to pass to the infinite volume limit $|\Lambda| \rightarrow \infty$. (This limit is also referred to as the thermodynamic limit). The Boltzmann-Gibbs distribution does not admit a direct extension to infinite systems. However, when dealing with infinite systems, we can still look at finite subsystems provided the rest is held fixed. Indeed, starting with an interacting potential ϕ we can define for each finite subsystem Λ a Hamiltonian Φ_Λ^ϕ which includes the interaction of Λ with its fixed environment. In the above, we argued that the physical systems like ferromagnets in equilibrium are reasonably modelled by Gibbs measures. We then should expect the Gibbs measure to exhibit a certain kind of behavior which reflects the physical phenomenon of phase transition. In order to find out what should happen, we consider the spontaneous magnetization of a ferromagnet at low temperature. First we place the ferromagnet in an external magnetic field (which oriented along one of the axes of the ferromagnetic crystal). Turning the field off and waiting until equilibrium, we find that the ferromagnet exhibits a macroscopic magnetic moment in the same direction as the stimulating external field. A second experiment with an external field in the opposite direction produces an equilibrium state with the opposite magnetization as before. The ferromagnet

thus admits two distinct equilibrium states. We thus expect that the physical phenomenon of phase transition should be reflected by the non-uniqueness of Gibbs measures. In 1968 Roland Dobrushin, [23], [24], [25] who is considered as one of the founders of modern rigorous Statistical Mechanics proposed a uniqueness condition which would imply the absence of phase transitions. The condition roughly stated that the total interaction of a given spin with all other spins should be very small. This has triggered some interest in the study of the exponential decay of the two-point correlation function. The study of the exponential decay of the correlation also gained much interest when Fröhlich and Spencer discovered in 1981 that the non-uniqueness of equilibrium state is not the only critical phenomenon of physical interest, but a different sort of transition is characterized by a change from an exponential decay of the correlation to a power law decay.

The methods for investigating phase transition for certain physical systems took an interesting direction when powerful and sophisticated PDE techniques are introduced in the mathematical technology. The methods are generally based on the analysis of suitable differential operators

$$\mathbf{W}_\Phi^{(0)} = \left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} \right)$$

and

$$\mathbf{W}_\Phi^{(1)} = -\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} + \mathbf{Hess}\Phi.$$

which are in some sense, deformations of the standard Laplace Beltrami operator. These operators commonly called Witten Laplacians were first introduced by Edward Witten [18] in 1982 in the context of Morse theory for the study of some topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Jöhanne Sjöstrand [8] introduced two elliptic differential operators.

$$A_\Phi^{(0)} := -\Delta + \nabla\Phi \cdot \nabla$$

and

$$A_\Phi^{(1)} := -\Delta + \nabla\Phi \cdot \nabla + \mathbf{Hess}\Phi$$

These later operators provide direct methods for the study of integrals and operators in high dimensions of the type that appear in Statistical Mechanics and Euclidean field theory. In 1996, Jöoperators are in fact equivalent to Witten's Laplacians. Since then, there has been significant advances in the use of these Laplacians to study the thermodynamic behavior of quantities related to the Gibbs measure $Z^{-1}e^{-\Phi}dx$. As a simple illustration, if one is interested in the study of the mean value $\langle g \rangle_\Lambda$ where

$$\langle g \rangle_\Lambda = \int g d\mu_\Lambda.$$

and

$$d\mu_\Lambda = \frac{e^{-\Phi_\Lambda} dx}{\int e^{\Phi_\Lambda} dx}$$

for a suitable smooth function g , one can first solve the equation

$$\nabla g = (-\Delta + \nabla\Phi \cdot \nabla) \mathbf{v} + \mathbf{Hess}\Phi \mathbf{v},$$

where the solution \mathbf{v} is a suitable C^∞ -vector field and the operator

$$-\Delta + \nabla\Phi \cdot \nabla$$

acts diagonally on each component of \mathbf{v} . Under certain assumptions on the Hamiltonian Φ one can see that \mathbf{v} is also a solution of the system

$$g = \langle g \rangle_\Lambda + \mathbf{v} \cdot \nabla\Phi - \operatorname{div}\mathbf{v}.$$

If it turns out that $g(0) = 0$ and 0 is a critical point of Φ then

$$\langle g \rangle_\Lambda = \operatorname{div}\mathbf{v}(0).$$

The study of the thermodynamic properties of the mean value is then reduced to estimating the derivatives of the solution \mathbf{v} .

Numerous techniques have been developed in the study of integrals associated to the equilibrium Gibbs state for certain unbounded spins systems. One of the most striking result is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one forms, leading to sophisticated methods for estimating the correlation functions. This formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [1]. The formula may be written as follow:

$$\mathbf{cov}(f, g) = \int \left(A_\Phi^{(1)^{-1}} \nabla f \cdot \nabla g \right) e^{-\Phi(x)} dx. \quad (1)$$

To understand the idea behind the formula mentioned above, let us denote by $\langle f \rangle$ the mean value of f with respect to the measure

$$e^{-\Phi(x)} dx,$$

the covariance of two functions f and g is defined by

$$\mathbf{cov}(f, g) = \langle (f - \langle f \rangle)(g - \langle g \rangle) \rangle \quad (2)$$

If one wants to have an expression of the covariance in the form

$$\mathbf{cov}(f, g) = \langle \nabla g \cdot \mathbf{w} \rangle_{L^2(\mathbb{R}^n, \mathbb{R}^n; e^{-\Phi} dx)}, \quad (3)$$

for a suitable vector field \mathbf{w} , we get, after observing that $\nabla g = \nabla(g - \langle g \rangle)$,

$$\mathbf{cov}(f, g) = \int (g - \langle g \rangle) (\nabla\Phi - \nabla) \cdot \mathbf{w} e^{-\Phi(x)} dx, \quad (4)$$

this leads to the question of solving the equation

$$f - \langle f \rangle = (\nabla\Phi - \nabla) \cdot \mathbf{w}. \quad (5)$$

Now trying to solve this above equation with $\mathbf{w} = \nabla u$, we obtain the equation

$$\left. \begin{aligned} f - \langle f \rangle &= A_{\Phi}^{(0)} u \\ \langle u \rangle &= 0 \end{aligned} \right\} \quad (6)$$

Assuming for now the existence of a smooth solution, we get by differentiation of this above equation

$$\nabla f = A_{\Phi}^{(1)} \nabla u \quad (7)$$

and the formula is now easy to see.

New methods that are purely based on spectral analysis have been recently developed by Helffer-Bodineau [2], Sjostrand-Bach-Jecko [26]. In these papers, the authors studied a certain class of unbounded spin models by means of the spectrum of the Witten Laplacian. In [26], the asymptotics of the two point correlation function to leading order in β^{-1} was obtained under weaker assumptions on the Hamiltonian. In 2003 V. Bach and J.S. Moller [27] proposed a refined version of the results in [26] by introducing a new twisted Witten Laplacian to relax the convexity assumptions.

We attempt in this paper, to study weighted estimates that lead to the exponential decay of the two-point correlation functions for certain convex unbounded spin systems. We removed limitations of earlier work of Helffer and Sjostrand [8]. They only treated the one dimensional case ($d = 1$) under the artificial restrictions

$$\|\mathbf{Hess}\Phi(x)\|_{\mathcal{L}(l_{\rho}^{\infty})} \leq C$$

and

$$\|\mathbf{Hess}\Phi(x) - \mathbf{I}\|_{\mathcal{L}(l_{\rho}^{\infty})} \leq \delta < 1,$$

for all weight function ρ on $\mathbb{Z}/m\mathbb{Z}$ satisfying

$$e^{-\kappa} \leq \frac{\rho(i+1)}{\rho(i)} \leq e^{\kappa}, \text{ for some } \kappa > 0.$$

These conditions are too restrictive for many important applications, while my conditions are considerably more flexible. In particular, the conditions in my work are suitable for treating the higher dimensional Kac model, where the potential is given by

$$\Phi(x) = \frac{x^2}{2} - \sum_{i,j \in \Lambda, i \sim j} \ln \cosh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right], \quad x = (x_i)_{i \in \Lambda},$$

for $\nu > 0$ small enough.

In section 2, we give a motivational background on the origin and nature of the Witten Laplacians. In section 3, we give an overview of the operators and equations in question. In section 4, we discuss preliminary results on Hilbert space methods for elliptic PDE's. In section 5, we provide a rigorous discussion based on Hilbert space methods for the solvability and regularity of the corresponding

Witten Laplacian equations. In section 6, we illustrate the family of Hamiltonians discussed in section 3 and 5 through an example of the type introduced by Marc Kac [20]. Section 7 is devoted to the study of the exponential decay of the two-point correlation functions for Kac-type models in the convex case.

2 The Witten's Laplacians

In 1982, Edward Witten published an article [18] on Supersymmetry and Morse theory relating invariants of a Riemannian manifold \mathbf{M} with some indices of a Morse function $\Phi \in C^\infty(\mathbf{M})$. For this he introduced the Witten derivative \mathbf{d}_Φ and the Witten coderivative \mathbf{d}_Φ^* by simply setting

$$\mathbf{d}_\Phi = e^{-\frac{\Phi}{2}} \mathbf{d} e^{\frac{\Phi}{2}} \quad \text{and} \quad \mathbf{d}_\Phi^* = e^{\frac{\Phi}{2}} \mathbf{d}^* e^{-\frac{\Phi}{2}}, \quad (8)$$

where \mathbf{d} and \mathbf{d}^* are the exterior derivative and exterior coderivative respectively. The Witten Laplacian is then defined to be the associated second order operator

$$\mathbf{W}_\Phi = (\mathbf{d}_\Phi + \mathbf{d}_\Phi^*)^2 \quad (9)$$

$$= \mathbf{d}_\Phi \mathbf{d}_\Phi^* + \mathbf{d}_\Phi^* \mathbf{d}_\Phi \quad (10)$$

acting on the exterior algebra bundle of the cotangent bundle of M as the standard Laplacian does.

Choosing a local orthonormal frame field $\mathbf{e}_1, \dots, \mathbf{e}_d$ and denoting by $\mathbf{e}^1, \dots, \mathbf{e}^d$ its dual coframe field, \mathbf{d} and \mathbf{d}^* could be easily represented in terms of the Riemannian connection ∇ as

$$\mathbf{d} = \mathbf{e}^i \wedge \nabla_{e_i} \quad \text{and} \quad \mathbf{d}^* = -\hat{i}_{e_j} \nabla_{e_j}. \quad (11)$$

where \hat{i}_{e_j} denote the interior product with respect to e_j (see [58] for more details). Here and in the rest of this section, we use the Einstein summation convention namely, an index occurring twice in a product is to be summed from 1 up to the space dimension. We consequently have

$$\mathbf{d}_\Phi = \mathbf{e}^i \wedge \nabla_{e_i} + \mathbf{e}^i \frac{\Phi_{;i}}{2} \quad \text{and} \quad \mathbf{d}_\Phi^* = -\mathbf{i}(e_j) \nabla_{e_j} + \mathbf{i}(e_j) \frac{\Phi_{;j}}{2} \quad (12)$$

where $\Phi_{;i_1 i_2 \dots}$ denote the components of multiple covariant differentiation relative to the local frame field $\mathbf{e}_1, \dots, \mathbf{e}_d$.

$$\Phi_{;ij} = \nabla_{e_j} \nabla_{e_i} \Phi - \nabla_{\nabla_{e_j} e_i} \Phi \quad (13)$$

Since $\mathbf{e}^i \wedge \nabla_{e_i}$ and $\mathbf{i}(e_j) \nabla_{e_j}$ do not depend on the choice of the local orthonormal frame and coframe field we may assume that $\mathbf{e}_1, \dots, \mathbf{e}_d$ comes from a normal coordinate centered at an arbitrary point and consequently have

$$\nabla_{e_j} \mathbf{e}^i \wedge = \nabla_{e_i} \mathbf{i}(e_j) = \mathbf{0}. \quad (14)$$

Now using (10), (11), (14) and the fact that

$$\mathbf{e}^i \wedge \mathbf{i}(\mathbf{e}_j) + \mathbf{i}(\mathbf{e}_j) \mathbf{e}^i \wedge = \delta_{ij}, \quad (15)$$

we have

$$\mathbf{W}_\Phi^{(p)} = \Delta + \frac{\Phi_{;i} \Phi_{;i}}{4} + \frac{\Phi_{;ij}}{2} (\mathbf{e}^i \wedge \mathbf{i}(\mathbf{e}_j) - \mathbf{i}(\mathbf{e}_j) \mathbf{e}^i \wedge). \quad (16)$$

In the case of \mathbb{R}^n where covariant differentiation becomes standard differentiation, the Witten Laplacian on 0-forms acting on a smooth function f gives

$$\mathbf{W}_\Phi^{(0)} f = -\Delta f + \frac{\Phi_{x_i} \Phi_{x_i}}{4} f - \frac{\Phi_{x_i x_i}}{2} f \quad (17)$$

$$= \left(-\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \right) f \quad (18)$$

The Witten Laplacian on one-forms acting on a one form

$$u = u^k(x) dx^k$$

gives

$$\mathbf{W}_\Phi^{(1)} u = \Delta u + \frac{\phi_{x_i} \phi_{x_i}}{4} u - \frac{\phi_{x_i x_i}}{2} u + 2 \frac{\phi_{x_k x_i}}{2} dx^i \wedge \mathbf{i} \frac{\partial}{\partial x^k} u \quad (19)$$

Identifying one-forms with vector fields in \mathbb{R}^n (1.12) becomes

$$\mathbf{W}_\Phi^{(1)} \mathbf{u} = \left(-\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \right) \otimes \mathbf{u} + \mathbf{Hess} \Phi \mathbf{u}. \quad (20)$$

Remark 1 *The tensor notation simply means that the operator $-\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2}$ acts diagonally on each component of the vector field \mathbf{u} . Let us also point out that the identification between forms and vector fields is a common practice in Riemannian geometry and is done via the metric tensor*

As first observed in [8] by Bernard Helffer and Johannes Sjöstrand, these Laplacians provide new methods for solving problems coming from Statistical Mechanics. The methods are generally based on the analysis of the differential operators

$$A_\Phi^{(0)} := -\Delta + \nabla \Phi \cdot \nabla \quad (21)$$

and

$$A_\Phi^{(1)} := A_\Phi^{(0)} \otimes Id + \mathbf{Hess} \Phi. \quad (22)$$

These two elliptic differential operators for which a Fredholm theory can be developed are equivalent, as observed in [13], to Witten's Laplacians $\mathbf{W}_\Phi^{(0)}$ and $\mathbf{W}_\Phi^{(1)}$ respectively where

$$\mathbf{W}_\Phi^{(0)} = -\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \quad (23)$$

and

$$\mathbf{W}_\Phi^{(1)} = \left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} \right) \otimes \mathbf{I} + \mathbf{Hess}\Phi. \quad (24)$$

Indeed, it only suffices to observe that

$$W_\Phi^{(\cdot)} = e^{-\Phi/2} \circ A_\Phi^{(\cdot)} \circ e^{\Phi/2} \quad (25)$$

and the map

$$\begin{aligned} U_\Phi & : L^2(\mathbb{R}^\Lambda) \rightarrow L^2(\mathbb{R}^\Lambda, e^{-\Phi} dx) \\ u & \mapsto e^{\frac{\Phi}{2}} u \end{aligned}$$

is unitary.

3 The Basic Equation

For any finite domain Λ of \mathbb{Z}^d , we shall consider a Hamiltonian Φ_Λ of the phase space \mathbb{R}^Λ satisfying conditions that will guaranty the solvability of the corresponding Witten Laplacian equations. We shall also consider a slowly growing source term g to ensure that the solutions have suitable asymptotic behavior.

We shall first establish the solvability of the equation

$$\begin{cases} A_\Phi^{(0)} f = g - \langle g \rangle_{L^2(\mu)} \\ \langle f \rangle_{L^2(\mu)} = 0 \end{cases} \quad (26)$$

by means of Hilbert space methods. The method consists of determining an appropriate function space and an operator which is a natural realization of the problem. In this particular problem, the function spaces to be considered are the Sobolev spaces $B_\Phi^k(\mathbb{R}^\Lambda)$ defined by

$$B_\Phi^k(\mathbb{R}^\Lambda) = \{ u \in L^2(\mathbb{R}^\Lambda) : Z_\Phi^l D^\alpha u \in L^2(\mathbb{R}^\Lambda) \forall l + |\alpha| \leq k \}.$$

where

$$Z_\Phi = \frac{|\nabla\Phi|}{2} \quad (27)$$

These are subspaces of the well known Sobolev spaces $W^{k,2}(\mathbb{R}^\Lambda)$, $k \in \mathbb{N}$. The vital tool in the Hilbert space approach to elliptic boundary value problems is the celebrated Lax-Milgram theorem. The essence of the method is the interpretation of the problem in a variational sense involving a bilinear form defined in a natural way by the problem and acting on the appropriately chosen function spaces.

In general, the Hilbert space method for elliptic differential equations uses the Compact embedding theorem for Sobolev spaces. This is a fundamental step in the method in order to be able to apply the Fredholm alternative. Since in the

context of our problem we are dealing with unbounded domains, the classical results regarding the compactness of the embedding

$$W^{k,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx) \quad (28)$$

for suitable Ω are no longer valid. However, In the case where the L^p spaces are taken with respect to the weighted measure $e^{-\Phi}dx$, with a suitable Φ , we have the following result due to J-M. Kneib and F. Mignot [11]lem.5

Lemma 2 *If Φ satisfies the condition*

$$\exists \theta \in (0, 1) : \lim_{|x| \rightarrow \infty} \theta |\nabla \Phi(x)|^2 - \Delta \Phi = \infty$$

then

$$H^1(\mu) \hookrightarrow L^2(\mathbb{R}^\Lambda, d\mu)$$

is compact.

Here and in the sequel, $d\mu$ will denote the Gibbs measure

$$d\mu = Z^{-1} e^{-\Phi} dx,$$

$$Z = \int_{\mathbb{R}^\Lambda} e^{-\Phi} dx.$$

and $H^k(\mu)$ denotes the weighted Sobolev space

$$H^k(\mu) = \{u \in L^2(\mathbb{R}^\Lambda, d\mu) : \partial^\alpha u \in L^2(\mathbb{R}^\Lambda, d\mu) \forall |\alpha| \leq k\}.$$

Proof. We shall prove that every bounded sequence in $H^1(\mu)$ has a convergent subsequence in $L^2(\mathbb{R}^\Lambda, d\mu)$. Let $\{u_k\} \subset H^1(\mu) = H^1(\mathbb{R}^\Lambda, d\mu)$ be such that

$$\|u_k\|_{H^1_\mu} \leq \sqrt{M} \quad \text{for every } k \quad \text{and some } M > 0.$$

For any $R > 0$, denote by $B(0, R)$ the open ball centered at 0 with radius R . It is clear that $H^1(\mathbb{R}^\Lambda, d\mu) \subset H^1(B(0, R), d\mu)$. Hence $\{u_k\}$ is a bounded sequence $H^1(B(0, R), d\mu)$. Moreover

$$\begin{aligned} & \int_{B(0,R)} u_k^2 dx + \int_{B(0,R)} |Du_k| dx \\ & \leq C_{\Phi,R} \left[\int_{B(0,R)} u_k^2 e^{-\Phi} dx + \int_{B(0,R)} |Du_k| e^{-\Phi} dx \right]. \end{aligned}$$

This implies that $\{u_k\}$ is a bounded sequence in $H^1(B(0, R))$. Now using the standard Sobolev compactness embedding theorem for bounded domains with nice boundary (see [3]), we get the compactness of the embedding

$$H^1(B(0, R)) \hookrightarrow L^2(B(0, R)).$$

Therefore, one can find a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that u_{k_j} converges in $L^2(B(0, R))$. We shall prove that $\{u_{k_j}\}$ is Cauchy in $L^2(\mathbb{R}^\Lambda, d\mu)$. Let $\eta > 0$. The assumption of the lemma implies that

$$\zeta := |\nabla\Phi|^2 - (1 + \eta)\Delta\Phi \quad (29)$$

is positive in a neighborhood of ∞ when $\theta = (1 + \eta)^{-1}$.

$$\begin{aligned} \int_{\mathbb{R}^\Lambda} |u_{k_j} - u_{k_l}|^2 e^{-\Phi} dx &\leq \int_{|x| < R} |u_{k_j} - u_{k_l}|^2 e^{-\Phi} + \int_{|x| \geq R} \frac{\zeta |u_{k_j} - u_{k_l}|^2}{\inf_{\mathbb{R}^\Lambda \setminus B(0, R)} \zeta} e^{-\Phi} dx \quad (30) \\ &\leq C_\Phi \int_{|x| < R} |u_{k_j} - u_{k_l}|^2 + \int_{|x| \geq R} \frac{\zeta |u_{k_j} - u_{k_l}|^2}{\inf_{\mathbb{R}^\Lambda \setminus B(0, R)} \zeta} e^{-\Phi} dx \quad (31) \end{aligned}$$

To estimate the last term of the right hand side of this last above inequality, let $\varepsilon > 0$ and choose R large enough so that

$$\inf_{\mathbb{R}^\Lambda \setminus B(0, R)} \zeta \geq \frac{4M(2 + \eta + \eta^{-1})}{\varepsilon}$$

Now Introduce the vector fields

$$X_j = \partial_j \quad (32)$$

and their formal adjoint in $L^2(\mu)$

$$X_j^* = -\partial_j + \Phi_{x_j}, \quad (33)$$

one has when $u \in C_0^\infty(\mathbb{R}^\Lambda)$ for their sum and commutator

$$(X_j + X_j^*) u = \Phi_{x_j} u \quad (34)$$

and

$$[X_j, X_j^*] u = \Phi_{x_j x_j} u \quad (35)$$

Now it is straightforward to see that

$$([X_j, X_j^*] u, u)_\mu = \|X_j^* u\|_\mu^2 - \|X_j u\|_\mu^2 \quad (36)$$

$$\|(X_j + X_j^*) u\|_\mu^2 \leq \left(1 + \frac{1}{\eta}\right) \|X_j u\|_\mu^2 + (1 + \eta) \|X_j^* u\|_\mu^2, \quad \forall \eta > 0 \quad (37)$$

so that a linear combination of these formulae gives for any $\eta > 0$

$$\left((|\nabla\Phi|^2 - (1 + \eta)\Delta\Phi) u, u \right)_\mu \leq (2 + \eta + \eta^{-1}) \left(\|X_1 u\|_\mu^2 + \dots + \|X_m u\|_\mu^2 \right). \quad (38)$$

Thus,

$$((\zeta u, u)_\mu \leq (2 + \eta + \eta^{-1}) \|u\|_{H^1(\mu)}^2 \quad (39)$$

Because $C_0^\infty(\mathbb{R}^\Lambda)$ is dense in $H^1(\mu)$, this inequality is valid for all $u \in H^1(\mu)$. Now applying (39) with u replaced by $u_{k_j} - u_{k_l}$, (31) gives

$$\begin{aligned} \int_{\mathbb{R}^\Lambda} |u_{k_j} - u_{k_l}|^2 e^{-\Phi} dx &\leq C_\Phi \int_{|x| < R} |u_{k_j} - u_{k_l}|^2 + \frac{(2 + \eta + \eta^{-1}) \|u_{k_j} - u_{k_l}\|_{H^1(\mu)}^2}{4M(2 + \eta + \eta^{-1})} \varepsilon \\ &\leq C_\Phi \int_{|x| < R} |u_{k_j} - u_{k_l}|^2 + \varepsilon \end{aligned}$$

The result follows from the convergence of the subsequence $\{u_{k_j}\}$ in $L^2(B(0, R))$.

■

The Lemma above indicates the direction towards the assumptions needed for the Hamiltonian $\Phi = \Phi_\Lambda$.

Assumptions on Φ .

Recall that Λ is a finite domain in \mathbb{Z}^d . We shall assume that $\Phi(x) \in C^\infty(\mathbb{R}^\Lambda)$ satisfying:

1. $\lim_{|x| \rightarrow \infty} |\nabla \Phi(x)| = \infty$
2. For some M , any $D^\alpha \Phi$ with $|\alpha| = M$ is bounded on \mathbb{R}^Λ .
3. For $|\alpha| \geq 1$, there are constants C_α such that $|D^\alpha \Phi(x)| \leq C_\alpha \left(1 + |\nabla \Phi(x)|^2\right)^{1/2}$
4. $\text{Hess} \Phi \geq \delta$ for some $0 < \delta \leq 1$

4 Preliminary Results on Hilbert Space Methods For Elliptic PDE

A bilinear form with domain H , a complex Hilbert space, is a complex-valued function a defined on $H \times H$ which is such that $a(u, v)$ is linear in u and conjugate linear in v . The inner product $(\cdot, \cdot)_H$ on H is clearly a sesquilinear form; we shall denote it by $1(\cdot, \cdot)$. The form $a + \lambda 1$ will simply be denoted by $a + \lambda$

$$(a + \lambda)(u, v) = a(u, v) + \lambda(u, v)_H$$

The adjoint a^* of a is defined by

$$a^*(u, v) = \overline{a(v, u)}$$

and a is said to be symmetric if $a \equiv a^*$, i.e. for all $u, v \in H$

$$a^*(u, v) = \overline{a(v, u)} = a(u, v).$$

A form is said to be bounded on $H \times H$ if there exists a constant $M > 0$ such that

$$|a(u, v)| \leq M \|u\|_H \|v\|_H \quad \text{for all } u, v \in H.$$

A form a is said to be coercive on H if there exists a positive constant $m > 0$ such that

$$|a(u, u)| \geq m \|u\|_H^2 \quad \text{for all } u, v \in H.$$

We shall say that a Banach space W is continuously embedded in a Banach space X if there is a bounded operator $E : W \rightarrow X$ which is one-to-one. We call E an embedding operator. We shall say that W is densely embedded in X if $R(E)$, the range of E is dense in X ; and we shall write

$$W \hookrightarrow_{ds}^E X.$$

If X is a Banach Space, the set of all linear conjugate functionals on X shall be denoted by X^* and is called the conjugate space of X^* .

Suppose X, Y, W, Z are Banach spaces such that

$$W \hookrightarrow_{ds}^E X \quad \text{and} \quad Y \hookrightarrow_{ds}^E Z^*.$$

Let $a(w, z)$ be a bounded bilinear form on $W \times Z$. We can define two linear operators connected with $a(w, z)$.

The first which we shall denote by A , is an operator from X to Y . We say that $x \in D(A)$, the domain of A and $Ax = y$ if $x \in R(E)$, $y \in Y$ and

$$a(E^{-1}x, z) = Fy(z), \quad \text{for all } z \in Z.$$

Since $R(F)$ is dense in Z^* , the operator A is well defined. We call A the operator associated with the bilinear form $a(u, v)$.

The second operator, which we denote by \hat{A} , is from W to Z^* . We define it as follows. Fix $w \in W$, $a(w, \cdot) \in Z^*$, it is bounded because the bilinear form a is bounded. We define $\hat{A}w$ to be $a(w, \cdot)$. \hat{A} is clearly well defined and will be called the extended linear operator associated with the bilinear form $a(u, v)$. It can be shown that A and \hat{A} are related in the following way:

$$A = F^{-1}\hat{A}E^{-1}$$

The fundamental tool to investigate the operator \hat{A} is the Lax-Milgram theorem

Theorem 3 (Lax Milgram) *Let a be a bounded coercive form on a Hilbert space H_o with bounds m and M as above. Then for any $F \in H_o^*$, the adjoint of H_o , there exists an $u \in H_o$ such that*

$$a(u, v) = \langle F, v \rangle \quad \text{for all } v \in H_o$$

The map $\hat{A} : u \mapsto F$ defined above is a linear bijection of H_o onto H_o^* and

$$m \leq \|\hat{A}\| \leq M, \quad M^{-1} \leq \|\hat{A}^{-1}\| \leq m^{-1}.$$

Proof. (see [3]). ■

Corollary 4 *For any choice of $F \in H_o^*$ there is a unique vector $u \in H_o$ satisfying*

$$(u, v)_{H_o} = F(v) \quad \text{for all } v \in H_o;$$

moreover, the isomorphism \hat{A}^{-1} from H_o^* onto H_o defined by $\hat{A}^{-1}F = u$ verifies

$$\|\hat{A}^{-1}F\|_{H_o} = \|F\|_{H_o^*}$$

Next, we apply the Lax-Milgram theorem to the situation where the Hilbert space H_o is continuously and densely embedded in another Hilbert H .

Lemma 5 *If H is a Hilbert space and W is a Banach space continuously and densely embedded in H with embedding operator E , then H can be continuously and densely embedded in W^* with embedding operator F satisfying*

$$(x, Ew)_H = Fx(w), \quad x \in H, \quad \text{and} \quad w \in W.$$

Proof. For each $x \in H$, the function $x^* : w \mapsto (x, Ew)_H$ is a conjugate linear functional on W and

$$|x^*(w)| \leq \|x\|_H \|E\| \|w\|_W.$$

Hence $x^* \in W^*$. Define the operator F from H to W^* by $Fx = x^*$. Clearly, F is linear and bounded. It is also one-to-one since $R(E)$ is dense in H . Finally, suppose $x^*(w) = 0$ for all $x^* \in R(F)$. Then $(x, Ew)_H = 0$ for all $x \in H$. Thus $Ew = 0$ and consequently $w = 0$. This shows that $R(F)$ is dense. ■

Now let the Hilbert space H_o be continuously and densely embedded into another Hilbert space H with embedding operator E . By the lemma above, H can be continuously and densely embedded in H_o^* with embedding operator F . We obtain the scheme

$$H_o \hookrightarrow_{ds}^E H \hookrightarrow_{ds}^F H_o^*$$

which is referred to by saying that (H_o, H, H_o^*) is a Hilbert triplet. Notice also that if the embedding E is compact, then so is the embedding

$$H_o \hookrightarrow^{FE} H_o^*.$$

Returning to the bilinear form on H_o , we weaken the notion of coerciveness as follows: We say that a bilinear form $a(u, v)$ on H_o is coercive relative to H , if there exists some $\lambda > 0$ such that $a_\lambda(u, v) = a(u, v) + \lambda(u, v)_H$ is coercive, i.e.

$$a(u, u) + \lambda \|u\|_H^2 \geq \alpha_o \|u\|_{H_o}^2 \quad \text{for } u \in H_o \text{ and some } \alpha_o > 0.$$

If this last inequality above holds, then by Lax-Milgram, the extended linear operator \hat{A}_λ associated with the bilinear form $a_\lambda(u, v)$ has a bounded inverse $\hat{A}_\lambda^{-1} : H_o^* \rightarrow H_o$, moreover $\hat{A}_\lambda u = \hat{A}u + \lambda \hat{B}u$, where \hat{A} is the extended operator associated with the bilinear form $a(u, v)$ and \hat{B} the extended operator associated with the inner product $(u, v)_H$.

Now Let $q \in H_o^*$ and consider the equation

$$u \in H_o, \quad \hat{A}u = q \tag{40}$$

(1.8) can now be written as

$$u \in H_o, \quad u - \lambda \hat{A}_\lambda^{-1} \hat{B}u = z \tag{41}$$

with $z = \hat{A}_\lambda^{-1} q$. We now claim that the compactness of the embedding E implies that of the operator $\hat{A}_\lambda^{-1} \hat{B} : H_o \rightarrow H_o$ is compact. Indeed this follows from

the fact that \hat{B} is bounded and $\hat{A}_\lambda^{-1} : H_o^* \rightarrow H_o$ is compact. By the Fredholm alternative (see theorem 3 below), (1.9) is uniquely solvable for any choice of $z \in H_o$ if and only if $u = 0$ is the unique vector of H_o satisfying $u - \lambda \hat{A}_\lambda^{-1} \hat{B}u = 0$. When this is the case, the linear operator $z \mapsto u$ defined by (1.9) is bounded from H_o to H_o . Summing up, we have the following theorem

Theorem 6 *Let (H_o, H, H_o^*) be a Hilbert triplet with H_o compactly embedded in H , let $a(u, v)$ be a bounded bilinear form on H_o coercive relative to H . Then*

$$u \in H_o, \quad a(u, v) = q(v) \quad \text{for } v \in H_o$$

admits a unique solution u for any choice of $q \in H_o^$ if and only if it admits a unique solution $u = 0$ for $q = 0$ in which case the solution u satisfies*

$$\|u\|_{H_o} \leq C \|q\|_{H_o}$$

with C dependent only on \hat{A}

Theorem 7 (Fredholm Alternative) *Let T be a compact linear operator on a Hilbert space V and consider the equations*

$$u \in V, \quad u - Tu = f \tag{42}$$

$$v \in V, \quad v^* - T^*v^* = g \tag{43}$$

where T^ the adjoint operator of T . Then the following alternative holds:*

- (i) either there exists a unique solution of (42) and (43) for any f and g in V , or*
- (ii) the homogeneous equation*

$$u - Tu = 0$$

has nontrivial solutions. In that case the dimension of the null space of $I - T$ is finite and equals the dimension of the null space \mathcal{N}^ of $I - T^*$. Furthermore (42) and (43) have solutions (not unique) if and only if*

$$\langle f, v^* \rangle = 0, \quad \forall v \in \mathcal{N}^*$$

and

$$\langle g, v \rangle = 0, \quad \forall v \in \mathcal{N}$$

\mathcal{N} being the null space of $I - T$

Proof. ..see Yosida [17],(X - §5) ■

Remark 8 *Assumption 2 implies that Φ is a slowly increasing function. This assumption is made to rule out any possibility of exponential growth for Φ .*

5 Sovability and Regularity of The Basic Equation

Theorem 9 *Let Λ be a finite domain in \mathbb{Z}^d . If Φ satisfies assumptions 1-4 above, then for any C^∞ -function g satisfying*

$$|D^\alpha g| \leq C_\alpha (1 + Z_\Phi)^{q_\alpha} \quad (44)$$

where

$$Z_\Phi = \frac{|\nabla\Phi|}{2},$$

$\alpha \in \mathbb{N}^{|\Lambda|}$ with some C_α and some $q_\alpha > 0$, there exists a unique C^∞ -function u solution of

$$\begin{cases} A_\Phi^{(0)} v = g - \langle g \rangle_{L^2(\mu)} \\ \langle v \rangle_{L^2(\mu)} = 0 \end{cases} \quad (45)$$

Proof. (Existence) We shall work in the unweighted space $L^2(\mathbb{R}^\Lambda)$ and with the Witten-Laplacians ensuing after the unitary transformation.

Under the unitary transformation,

$$A_\Phi^{(0)} v = g - \langle g \rangle_{L^2(\mu)} \quad \text{in } \mathbb{R}^\Lambda$$

is equivalent to

$$\mathbf{W}_\Phi^{(0)} u = q \quad \text{in } \mathbb{R}^\Lambda$$

where

$$u = e^{-\Phi/2} v \quad \text{and} \quad q = e^{-\Phi/2} (g - \langle g \rangle_{L^2(\mu)}) \in L^2(\mathbb{R}^\Lambda)$$

First observe that $\mathbf{W}_\Phi^{(0)}$ is a positive unbounded operator acting on $L^2(\mathbb{R}^\Lambda)$; indeed

$$\mathbf{W}_\Phi^{(0)} = \left(\partial_x + \frac{\nabla\Phi}{2} \right)^* \left(\partial_x + \frac{\nabla\Phi}{2} \right)$$

where

$$\left(\partial_x + \frac{\nabla\Phi}{2} \right)^* = \left(-\partial_x + \frac{\nabla\Phi}{2} \right)$$

Let

$$B_\Phi^k(\mathbb{R}^\Lambda) = \{u \in L^2(\mathbb{R}^\Lambda) : Z_\Phi^l D^\alpha u \in L^2(\mathbb{R}^\Lambda) \forall l + |\alpha| \leq k\}$$

(here $D^\alpha u$ is taken in the distributional sense in \mathbb{R}^Λ).

Denote by $B_{o,\Phi}^1(\mathbb{R}^\Lambda)$ be the closure of $C_o^\infty(\mathbb{R}^\Lambda)$ in $B_\Phi^1(\mathbb{R}^\Lambda)$, and let \mathbf{b} be the bilinear form on $B_{o,\Phi}^1(\mathbb{R}^\Lambda)$ defined by

$$\mathbf{b} : B_{o,\Phi}^1(\mathbb{R}^\Lambda) \times B_{o,\Phi}^1(\mathbb{R}^\Lambda) \rightarrow \mathbb{R}$$

with

$$\mathbf{b}(u, w) = \int_{\mathbb{R}^\Lambda} Du \cdot Dw dx + \int_{\mathbb{R}^\Lambda} \left(\frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} \right) u w dx$$

Because we have in mind to apply theorems 6 and 7 above, we need to check boundedness and coerciveness of \mathbf{b} .

Boundedness: After observing that

$$\begin{aligned}\Delta\Phi &\leq C(1 + |\nabla\Phi|^2)^{1/2} \\ &\leq C(1 + |\nabla\Phi|^2),\end{aligned}$$

it then follows immediately from Cauchy-Schwartz inequality that

$$|\mathbf{b}(u, w)| \leq \alpha_o \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)} \|w\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}$$

for some constant $\alpha_o > 0$.

Coerciveness:

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} |Du|^2 dx &= \mathbf{b}(u, u) - \int_{\mathbb{R}^\Lambda} \left(\frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} \right) |u|^2 dx \\ \int_{\mathbb{R}^\Lambda} |Du|^2 dx + \int_{\mathbb{R}^\Lambda} |Z_\Phi u|^2 dx &= \mathbf{b}(u, u) + \int_{\mathbb{R}^\Lambda} \frac{\Delta\Phi}{2} |u|^2 dx \\ &\leq \mathbf{b}(u, u) + \varepsilon \int_{\mathbb{R}^\Lambda} \frac{(\Delta\Phi)^2}{4} |u|^2 dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}^\Lambda} |u|^2 dx \\ &\leq \mathbf{b}(u, u) + C\varepsilon \int_{\mathbb{R}^\Lambda} |Z_\Phi u|^2 dx + \left(C\varepsilon + \frac{1}{4\varepsilon} \right) \int_{\mathbb{R}^\Lambda} |u|^2 dx\end{aligned}$$

choosing ε such that $C\varepsilon < 1$ and adding $\int_{\mathbb{R}^\Lambda} |u|^2 dx$ on both side of this above inequality, we immediately get

$$\delta \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 \leq \mathbf{b}(u, u) + \gamma \|u\|_{L^2(\mathbb{R}^\Lambda)}^2 \quad (46)$$

for some positive constants δ and γ .

This shows that the bilinear form $\mathbf{b}(u, v)$ is bounded and coercive relative to $L^2(\mathbb{R}^\Lambda)$.

Observe that $B_{o,\Phi}^1(\mathbb{R}^\Lambda)$ is densely embedded into $L^2(\mathbb{R}^\Lambda)$. Now considering the Hilbert triplet

$$\left(B_{o,\Phi}^1(\mathbb{R}^\Lambda), L^2(\mathbb{R}^\Lambda), B_{o,\Phi}^{-1}(\mathbb{R}^\Lambda) \right), \quad (47)$$

where $B_{o,\Phi}^{-1}(\mathbb{R}^\Lambda)$ denote the conjugate space of $B_{o,\Phi}^1(\mathbb{R}^\Lambda)$.

We need to check that the embedding

$$B_{o,\Phi}^1(\mathbb{R}^\Lambda) \hookrightarrow L^2(\mathbb{R}^\Lambda)$$

is compact. This follows from Lemma 1 by simply observing that

$$B_{o,\Phi}^1(\mathbb{R}^\Lambda) \subset U_\Phi^{-1}(H^1(\mu))$$

and the fact that U_Φ is a unitary operator.
Let \mathbf{B}_γ be the bilinear form in $B_{o,\Phi}^1(\mathbb{R}^\Lambda)$ defined by

$$\mathbf{B}_\gamma(u, w) = \mathbf{b}(u, w) + \gamma \langle u, w \rangle_{L^2(\mathbb{R}^\Lambda)}$$

Let

$$\hat{A}_\gamma : B_{o,\Phi}^1(\mathbb{R}^\Lambda) \rightarrow B_{o,\Phi}^{-1}(\mathbb{R}^\Lambda)$$

be the extended linear operator associated with the bilinear form $\mathbf{B}_\gamma(u, w)$. We have

$$\hat{A}_\gamma u = \hat{A}u + \gamma \hat{B}u, \quad (48)$$

where \hat{A} and \hat{B} are the bounded bilinear forms associated with \mathbf{b} and $(\cdot, \cdot)_{L^2}$ respectively.

Note that the equation

$$u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda) \quad \hat{A}u = q$$

is the variational interpretation of the equation

$$\mathbf{W}_\Phi^{(0)} u = q \quad \text{in } \mathbb{R}^\Lambda.$$

By theorem 1 (Lax-Milgram), the boundedness of \mathbf{B}_γ and the coercivity condition

$$\mathbf{B}_\gamma(u, u) \geq \delta \|u\|_{B^1(\mathbb{R}^\Lambda)}^2 \quad \forall u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda)$$

guarantee that A_γ has a bounded inverse

$$\hat{A}_\gamma^{-1} : B_{o,\Phi}^{-1}(\mathbb{R}^\Lambda) \rightarrow B_{o,\Phi}^1(\mathbb{R}^\Lambda).$$

Now using the fact that

$$\hat{A}_\gamma u = \hat{A}u + \gamma \hat{B}u,$$

we can write the equation

$$u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda) \quad \hat{A}u = q$$

as

$$u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda), \quad u - \gamma \hat{A}_\gamma^{-1} \hat{B}u = z \quad (49)$$

where

$$z = \hat{A}_\gamma^{-1} q \quad (50)$$

As in the preliminary, because the injection

$$B_{o,\Phi}^1(\mathbb{R}^\Lambda) \hookrightarrow L^2(\mathbb{R}^\Lambda)$$

is compact, the operator $\gamma \hat{A}_\gamma^{-1} \hat{B} : B_{o,\Phi}^1(\mathbb{R}^\Lambda) \rightarrow B_{o,\Phi}^1(\mathbb{R}^\Lambda)$. Moreover, the boundedness of $\gamma \hat{A}_\gamma^{-1} \hat{B}$ implies that

$$\left(\gamma \hat{A}_\gamma^{-1} \hat{B}\right)^* = \left(\gamma \left(\hat{B}_\gamma^{-1} \hat{A}_\gamma\right)^*\right)^{-1} \quad (51)$$

$$= \gamma \left(\hat{A}_\gamma^* \left(\hat{B}^{-1}\right)^*\right)^{-1} \quad (52)$$

$$= \gamma \left(\hat{A}_\gamma^* \left(\hat{B}^*\right)^{-1}\right)^{-1} \quad (53)$$

$$= \gamma \hat{A}_\gamma^{-1} \hat{B} \quad (54)$$

Observe that the equality above is not necessarily true in the case of unbounded operators because one would first have to check that the inverse is densely defined for the existence of the adjoint. Let us also point out that the self-adjointness of \hat{A}_γ and \hat{B} follow from the fact that they are both associated with symmetric bilinear forms.

Now observe that

$$\ker(I - \gamma \hat{A}_\gamma^{-1} \hat{B}) \subset \ker A \quad (55)$$

We now claim that

$$\ker \hat{A} = \left\{ \delta e^{-\Phi/2}, \delta \in \mathbb{R} \right\}. \quad (56)$$

Indeed if $\hat{A}u = 0$, then $\mathbf{b}(u, u) = 0$. Hence

$$\left\| \left(\partial_x + \frac{\nabla \Phi}{2} \right) u \right\|_{L^2}^2 = 0$$

which would imply that u is a solution of the equation

$$\left(\partial_x + \frac{\nabla \Phi}{2} \right) u = 0$$

One can then easily see u must be a constant multiple of $e^{-\Phi/2}$. We have in mind to apply the second part of Theorem 7 (Fredholm alternative). This brings us to check orthogonality of q with $\ker(I - \gamma \hat{A}_\gamma^{-1} \hat{B})$. Let $\delta \in \mathbb{R}$,

$$\left\langle \delta e^{-\Phi/2}, q \right\rangle_{L^2(\mathbb{R}^\Lambda)} = \int_{\mathbb{R}^\Lambda} \delta e^{-\Phi/2} e^{-\Phi/2} (g - \langle g \rangle_{L^2(\mu)}) \quad (57)$$

$$= \delta \left(\langle g \rangle_{L^2(\mu)} - \langle g \rangle_{L^2(\mu)} \right) = 0 \quad (58)$$

Hence using part (ii) of theorem 3, we conclude that the equation

$$\hat{A}u = q \quad (59)$$

is solvable therefore

$$A_\Phi^{(0)} v = g - \langle g \rangle_{L^2(\mu)} \quad (60)$$

is solvable in the weak sense. To complete the proof of theorem 4, we need to prove that the L^2 -solution constructed above is a classical solution ■

Regularity: Next, we shall prove that the weak solution constructed above is actually a classical solution. The proof is based on the method of difference quotient.

Theorem 10 (B^k -regularity) *Given $q \in B_{\Phi}^{k-1}(\mathbb{R}^\Lambda)$ for $k = 0, 1, 2, \dots$, a solution $u \in B_{o, \Phi}^1(\mathbb{R}^\Lambda)$ of*

$$\hat{A}u = q \quad (61)$$

is an element of $B_{\Phi}^{k+1}(\mathbb{R}^\Lambda)$ and we have the estimate

$$\|u\|_{B_{\Phi}^{k+1}(\mathbb{R}^\Lambda)} \leq C \left[\|\hat{A}u\|_{B_{\Phi}^{k-1}(\mathbb{R}^\Lambda)} + \|u\|_{B_{\Phi}^k(\mathbb{R}^\Lambda)} \right] \quad (62)$$

for all $u \in B_{\Phi}^{k+1}(\mathbb{R}^\Lambda)$.

Proof. We first establish the result when $k = 0$.
We have

$$\left(\frac{\Delta \Phi}{2} u, u \right)_{L^2} \leq \left\| \frac{\Delta \Phi}{2} u \right\|_{L^2(\mathbb{R}^\Lambda)} \|u\|_{L^2(\mathbb{R}^\Lambda)} \quad (63)$$

$$\leq C \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)} \|u\|_{L^2(\mathbb{R}^\Lambda)} \quad (64)$$

$$\leq \varepsilon C \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 + \frac{C}{4\varepsilon} \|u\|_{L^2(\mathbb{R}^\Lambda)}^2 \quad (65)$$

Thus, for $u \in B_{o, \Phi}^1(\mathbb{R}^\Lambda)$, we have

$$\begin{aligned} \langle \hat{A}u, u \rangle &= \|Du\|_{L^2(\mathbb{R}^\Lambda)}^2 + (Z_{\Phi}^2 u, u)_{L^2} - \left(\frac{\Delta \Phi}{2} u, u \right)_{L^2} \\ &\geq \|Du\|_{L^2(\mathbb{R}^\Lambda)}^2 + \|Z_{\Phi} u\|_{L^2(\mathbb{R}^\Lambda)}^2 - \varepsilon C \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 - \frac{C}{4\varepsilon} \|u\|_{L^2(\mathbb{R}^\Lambda)}^2. \end{aligned}$$

Choosing ε such that $\varepsilon C < 1$, we get

$$\langle \hat{A}u, u \rangle \geq C \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 - C \|u\|_{L^2(\mathbb{R}^\Lambda)}^2$$

Hence

$$\begin{aligned} \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 &\leq C \langle \hat{A}u, u \rangle + C \|u\|_{L^2(\mathbb{R}^\Lambda)}^2 \\ &\leq C \left\| \hat{A}u \right\|_{B_{\Phi}^{-1}(\mathbb{R}^\Lambda)} \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)} + C \|u\|_{L^2(\mathbb{R}^\Lambda)}^2 \\ &\leq \frac{C}{4\varepsilon} \left\| \hat{A}u \right\|_{B_{\Phi}^{-1}(\mathbb{R}^\Lambda)}^2 + C\varepsilon \|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 + C \|u\|_{L^2(\mathbb{R}^\Lambda)}^2 \end{aligned}$$

Again choosing ε appropriately, ($\varepsilon C < 1$) we finally get

$$\|u\|_{B_{\Phi}^1(\mathbb{R}^\Lambda)}^2 \leq C \left\| \hat{A}u \right\|_{B_{\Phi}^{-1}(\mathbb{R}^\Lambda)}^2 + C \|u\|_{B_{\Phi}^0(\mathbb{R}^\Lambda)}^2.$$

Now assume that for $u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda)$, $\hat{A}u = q \in B_\Phi^{k-1}(\mathbb{R}^\Lambda)$ implies $u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$ and that

$$\|u\|_{B_\Phi^{k+1}(\mathbb{R}^\Lambda)} \leq C \left[\|\hat{A}u\|_{B_\Phi^{k-1}(\mathbb{R}^\Lambda)} + \|u\|_{B_\Phi^k(\mathbb{R}^\Lambda)} \right] \quad (66)$$

Suppose now that $u \in B_{o,\Phi}^1(\mathbb{R}^\Lambda)$, $\hat{A}u \in B_\Phi^k(\mathbb{R}^\Lambda)$. So we know that $u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$ and we want to establish that $u \in B_\Phi^{k+2}(\mathbb{R}^\Lambda)$.

Because

$$D_i^h u = \frac{u(x + he_i) - u(x)}{h} \in B_\Phi^{k+1}(\mathbb{R}^\Lambda),$$

replacing u by $D_i^h u$ in inequality (66) we get

$$\begin{aligned} \|D_i^h u\|_{B_\Phi^{k+1}(\mathbb{R}^\Lambda)} &\leq C \left[\|\hat{A}D_i^h u\|_{B_\Phi^{k-1}(\mathbb{R}^\Lambda)} + \|D_i^h u\|_{B_\Phi^k(\mathbb{R}^\Lambda)} \right] \\ &\leq C \left[\|D_i^{-h} \hat{A}u\|_{B_\Phi^{k-1}(\mathbb{R}^\Lambda)} + \|uD_i^h X_\Phi\|_{B_\Phi^{k-1}(\mathbb{R}^\Lambda)} + \|D_i^h u\|_{B_\Phi^k(\mathbb{R}^\Lambda)} \right] \end{aligned}$$

where

$$X_\Phi := \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}$$

Now letting $h \rightarrow 0$ and using assumption 3 on Φ we get

$$\|D_i u\|_{B_\Phi^{k+1}(\mathbb{R}^\Lambda)} \leq C \left[\|\hat{A}u\|_{B_\Phi^k(\mathbb{R}^\Lambda)} + \|u\|_{B_\Phi^k(\mathbb{R}^\Lambda)} + \|u\|_{B_\Phi^{k+1}(\mathbb{R}^\Lambda)} \right]$$

it then follows that

$$D_i u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda).$$

It then only remains to prove that $Z_\Phi^{k+2}u \in L^2(\mathbb{R}^\Lambda)$. To see this first observe that

$$Z_\Phi^2 u = \hat{A}u + \Delta u + \frac{\Delta\Phi}{2}u \quad (67)$$

Here, the Laplacian is taken in the distributional sense. Multiplying by Z_Φ^k on both sides of this last equality, we obtain:

$$Z_\Phi^{k+2}u = Z_\Phi^k \hat{A}u + Z_\Phi^k \Delta u + Z_\Phi^k \frac{\Delta\Phi}{2}u. \quad (68)$$

The first term of this equality is in $L^2(\mathbb{R}^\Lambda)$ because $\hat{A}u \in B_\Phi^k(\mathbb{R}^\Lambda)$. That the second terms also belongs to $L^2(\mathbb{R}^\Lambda)$ follows from the fact that $D_i u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$. Finally to see that the last term is an element of $L^2(\mathbb{R}^\Lambda)$, we use assumption 3 on Φ to get that

$$\frac{\Delta\Phi}{2} \leq C \left(\frac{1}{4} + Z_\Phi^2 \right)^{1/2} \quad (69)$$

$$\leq C \left(\frac{1}{2} + Z_\Phi \right), \quad (70)$$

and use the fact that $u \in B_\Phi^{k+1}(\mathbb{R}^\Lambda)$ ■

Proposition 11 (C^∞ -regularity) *The weak solution u of $W_\Phi^{(0)}u = q$ is an element of $C^\infty(\mathbb{R}^\Lambda)$.*

The proof of this proposition use the general sobolev inequalities theorem given below.

Theorem 12 (General Sobolev Inequality) *Let U be a bounded open subset of \mathbb{R}^n , with a C^1 -boundary. Assume $u \in W^{k,p}(U)$ where*

$$W^{k,p}(U) := \{u \in L^1_{loc}(\mathbb{R}^n) : \partial^\alpha u \in L^p(\mathbb{R}^n) \quad \forall |\alpha| \leq k\}$$

If

$$k > \frac{n}{p}$$

then $u \in C^{k - [\frac{n}{p}] - 1, \gamma}(\bar{U})$, where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer.} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

Here $C^{k,\alpha}(\bar{U})$ is the Hölder space consisting of all functions $u \in C^k(\bar{U})$ such that

$$\|u\|_{C^{k,\alpha}(\bar{U})} := \sum_{|\beta| \leq k} \sup_{x \in U} |D^\beta u(x)| + \sum_{|\beta|=k} \sup_{\substack{x,y \in U \\ x \neq y}} \left| \frac{D^\beta u(x) - D^\beta u(y)}{|x-y|^\alpha} \right| < \infty$$

Proof. see [3] ■

Proof of proposition 11. Because $q \in C^\infty(\mathbb{R}^\Lambda)$, we have $u \in B^k_{loc}(\mathbb{R}^\Lambda) \forall k$, which implies $u \in H^k(V) (= W^{k,2}(V)) \forall k$ and $\forall V \subset\subset \mathbb{R}^\Lambda$. Now choose $k \in \mathbb{N}$ such that $k > |\Lambda|$. Then the theorem above implies that $u \in C^{k,\gamma}(\bar{V})$ for some $0 < \gamma < 1$. Consequently, $u \in C^k(V)$ for an arbitrary big enough k and for any $V \subset\subset \mathbb{R}^\Lambda$ ■

Now that we have enough smoothness, we can make the following remark which completes the proof of theorem 9.

Remark 13 *A simple integration by parts argument shows that u is in fact a strong solution. It satisfies*

$$W_\Phi^{(0)}u = q$$

pointwise almost everywhere. Using the unitary transformation and taking gradient on both sides of

$$A_\Phi^{(0)}v = g - \langle g \rangle_{L^2(\mu)},$$

we get

$$A_\Phi^{(1)}\nabla v = \nabla g.$$

If \mathbf{q} is a smooth vector field satisfying

$$|\partial^\alpha \mathbf{q}| \leq C_\alpha (1 + Z_\Phi)^{q_\alpha} \quad \text{for some } q_\alpha > 0, \quad (71)$$

then one can show as above (this time using uniqueness result of the Fredholm alternative) that the equation

$$A_{\Phi}^{(1)} \mathbf{v} = \mathbf{q}$$

has a unique weak solution.

$A_{\Phi}^{(1)} \nabla v = \nabla g$ would then imply that two solutions of

$$A_{\Phi}^{(0)} v = g - \langle g \rangle_{L^2(\mu)} \tag{72}$$

must differ by a constant. Thus the problem

$$\begin{cases} A_{\Phi}^{(0)} u = g - \langle g \rangle_{L^2(\mu)} \\ \langle u \rangle_{L^2(\mu)} = 0 \end{cases}$$

has a unique solution. This ends the proof of theorem 9

6 The Kac-like Model

In this section, we propose to illustrate the results above through the study of a more specific family of classical unbounded spin model related to statistical mechanics and is given by

$$\Phi(x) = \Phi_{\Lambda}(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^{\Lambda}. \tag{73}$$

Here we have used the notation $x^2 = x \cdot x$.

The model that was originally suggested by M. Kac and corresponds to when Ψ is given by

$$\Psi(x) = - \sum_{i,j \in \Lambda, i \sim j} \ln \cosh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]$$

where ν is a small positive constant.

Other aspects of this family of potentials is studied in [8] in the one dimensional case

Definition 14 *The lattice support, S_g of a function g on \mathbb{R}^{Λ} is defined to be the smallest subset Γ of Λ for which g can be written as function of x_l alone with $l \in \Gamma$. For instance, if $g = x_i$, $S_g = \{i\}$.*

Under the assumptions

$$|\partial^{\alpha} \nabla \Psi| \leq C_{\alpha}, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{74}$$

$$\mathbf{Hess} \Phi \geq \delta > 0, \quad 0 < \delta < 1, \tag{75}$$

One can check that Φ satisfies the assumptions 1-4 in section 3.

Let g be a smooth function on \mathbb{R}^Γ where Γ is a fixed subset. We shall use the notation

$$x_\Sigma = (x_i)_{i \in \Sigma}$$

if Σ is a proper subset of Λ and shall also assume that $S_g = \Gamma$. Now define the function \tilde{g} on \mathbb{R}^Λ by

$$\tilde{g}(x) = g(x_\Gamma), \quad x \in \mathbb{R}^\Lambda.$$

If there is no ambiguity we shall identify \tilde{g} with g .

We propose to prove that if in addition to the assumptions above on Φ , the functions Ψ and g are compactly supported and g satisfies,

$$|\partial^\alpha \nabla g| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|},$$

then the solution v of the equation

$$\begin{cases} -\Delta v + \nabla \Phi \cdot \nabla v = g - \langle g \rangle_{L^2(\mu)} & \text{in } \mathbb{R}^\Lambda \\ \langle v \rangle_{L^2(\mu)} = 0 \end{cases} \quad (76)$$

constructed in section 5 satisfies

$$\partial^\alpha \nabla v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad \forall \alpha \in \mathbb{N}^{|\Lambda|} \quad (77)$$

Recall that under a suitable change of variables, the equation

$$A_\Phi^{(1)} v = \nabla g \quad (78)$$

could be written as

$$\left(-\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \right) \otimes \mathbf{u} + \mathbf{Hess} \Phi \mathbf{u} = \mathbf{q} \quad (79)$$

where

$$\mathbf{u} = e^{-\Phi/2} \nabla v \quad \text{and} \quad \mathbf{q} = e^{-\Phi/2} \nabla g \quad (80)$$

Let $B_1 = B_{R_1}(0) \subset \mathbb{R}^\Lambda$ denote a large balls centered at zero with radius R_1 and containing the support of Ψ in \mathbb{R}^Λ . We also consider a ball $B_2 = B_{R_2}(0) \subset \mathbb{R}^\Gamma$ of radius $R_2 > R_1$ containing the support of g in \mathbb{R}^Γ . The support of \tilde{g} in \mathbb{R}^Λ is then contained in the cylinder

$$B = B_2 \times \mathbb{R}^{\Lambda \setminus \Gamma}$$

In $B^c = \mathbb{R}^\Lambda \setminus B$ we have

$$\begin{cases} \left(-\Delta + \frac{x^2}{4} - \frac{m}{2} + \mathbf{I} \right) \mathbf{u} = 0 & \text{in } B^c \\ \mathbf{u} = \varphi & \text{on } \partial B \text{ (in the trace sense)} \end{cases} \quad (81)$$

Here φ is a C^∞ -vector field on ∂B and $m = |\Lambda|$.

Since the operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + \mathbf{I} \quad (82)$$

acts diagonally on u , we can work component by component and the situation is reduced to the scalar case

$$\begin{cases} \left(-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1\right) u = 0 & \text{in } B^c \\ u = \varphi & \text{on } \partial B \text{ (in the trace sense)} \end{cases}. \quad (83)$$

Having reduced the problem to a Dirichlet type for the Schrodinger operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1, \quad (84)$$

we shall need some results on the decay of eigenfunctions of the corresponding Schrodinger operator. We need the following lemma:

Lemma 15 *The fundamental solution $\mathcal{E} \in \mathcal{S}'(\mathbb{R}^\Lambda)$ of the operator $-\Delta + k^2$ ($k > 0$) exists and is unique. It is spherically symmetric, is an element of $C^\infty(\mathbb{R}^\Lambda \setminus \{0\})$ and has the following asymptotics as $|x| \rightarrow \infty$:*

$$\mathcal{E}(x) = C |x|^{\left(\frac{m-1}{2}\right)} e^{-k|x|} (1 + o(1)) \quad (85)$$

In the Lemma, $\mathcal{S}'(\mathbb{R}^\Lambda)$ denotes the space of tempered distributions on \mathbb{R}^Λ

Proof. Consider the equation

$$(-\Delta + k^2) \mathcal{E}(x) = \delta_o(x). \quad (86)$$

Taking Fourier transform, we get

$$(-\Delta + \widehat{k^2}) \widehat{\mathcal{E}}(x) = \widehat{\delta_o(x)}. \quad (87)$$

equivalently

$$(x^2 + k^2) \widehat{\mathcal{E}}(x) = (2\pi)^{-m/2} \quad (88)$$

which implies

$$\widehat{\mathcal{E}}(x) = \frac{(2\pi)^{-m/2}}{x^2 + k^2} \quad (89)$$

The uniqueness and spherical symmetry follow since

$$\mathcal{E}(x) = (2\pi)^{-m/2} \widehat{\widehat{\mathcal{E}}(x)}. \quad (90)$$

Furthermore, if $x \neq 0$, the smoothness of $\mathcal{E}(x)$ follows from the regularity theory of the elliptic equation as discussed above in section 5.

$$(-\Delta + k^2) \mathcal{E}(x) = 0 \text{ in } \mathbb{R}^\Lambda \setminus \{0\} \quad (91)$$

for $x \neq 0$ set $\mathcal{E}(x) = f(r)$ where $f \in C^\infty(\mathbb{R}^+)$ and $r = |x|$. (91) becomes

$$-f''(r) - \frac{m-1}{r} f'(r) + k^2 f(r) = 0 \quad (92)$$

Set $f(r) = a(r)g(r)$. Plugging this in (92) and setting the coefficient of $g'(r)$ equal zero gives

$$2a' + \frac{m-1}{r}a = 0 \quad (93)$$

Take

$$a(r) = r^{-\frac{m-1}{2}}.$$

Then

$$f(r) = r^{-\frac{m-1}{2}}g(r)$$

and (92) takes the form

$$g''(r) - \zeta^2(x)g(r) = 0 \quad (94)$$

where $\zeta(x) = \sqrt{k^2 + \frac{C}{r^2}}$. (C is a positive constant depending only on m which we assume > 3). Now using classical results on the asymptotics of the solutions of the one dimensional schrodinger equation (see Appendix), we discover that the asymptotics of the solutions of (94) are given by

$$g_{\pm}(r) = \frac{1}{\sqrt{\sqrt{k^2 + \frac{C}{r^2}}}} e^{\pm \int_{r_0}^r \sqrt{k^2 + \frac{C}{s^2}} ds} (1 + o(1)) \quad (95)$$

Since

$$1/\sqrt{\sqrt{k^2 + \frac{C}{r^2}}} = C + o\left(\frac{1}{r}\right), \quad \int_{r_0}^r \sqrt{k^2 + \frac{C}{s^2}} ds = r + C + o(1) \quad (96)$$

Now using the fact that

$$e^{o(1)} = 1 + o(1), \quad (97)$$

we have

$$g_{\pm}(r) = Ce^{\pm kr} (1 + o(1)). \quad (98)$$

Hence the asymptotics of the solutions of (94) are

$$f_{\pm}(r) = Cr^{-\frac{m-1}{2}} e^{\pm kr} (1 + o(1)). \quad (99)$$

Since $\mathcal{E}(x) = f(|x|) \in \mathcal{S}'(\mathbb{R}^{\Lambda})$, we conclude that $f = f_-$ and the result follows

■

Theorem 16 *Let Ω be any exterior domain in \mathbb{R}^{Λ} containing a neighborhood of infinity with smooth internal boundary. Let the potential $v(x) \in C^{\infty}(\Omega)$ and satisfy*

$$\liminf_{|x| \rightarrow \infty} v(x) \geq E \quad (100)$$

and let φ be a smooth solution of the problem

$$\begin{cases} (-\Delta + v(x)) \varphi = \lambda \varphi & \text{in } \Omega \\ \rho = \psi & \text{on } \partial\Omega \end{cases} \quad (101)$$

where $\lambda < E$ and φ is a smooth function on $\partial\Omega$. Then the following estimate holds:

$$|\varphi(x)| \leq C_\varepsilon e^{-\sqrt{(a-\lambda-\varepsilon)/2}|x|} \quad (102)$$

for any $\varepsilon > 0$

The proof of this theorem uses the following lemma

Lemma 17 (A Maximum principle) *Let $k > 0$, Σ an open subset of \mathbb{R}^Λ , and $u \in C^2(\Sigma)$ a function such that*

$$(-\Delta + k^2) u = f \leq 0 \quad \text{in } \Sigma. \quad (103)$$

Then u cannot have a positive maximum in Σ .

Proof. If $x_o \in \Sigma$ is a maximum point and $u(x_o) > 0$, then

$$\Delta u(x_o) \leq 0; \quad (104)$$

this contradicts (103). ■

Proof of Theorem 7. Let φ be a real solution of the equation

$$H\varphi = \lambda\varphi \quad \text{in } \Omega. \quad (105)$$

where

$$H = -\Delta + v(x)$$

We obviously have

$$\Delta(\varphi^2) = 2\Delta\varphi \cdot \varphi + 2|\nabla\varphi|^2 \quad (106)$$

$H\varphi = \lambda\varphi$ gives $-\Delta\varphi = (\lambda - v(x))\varphi$ which implies

$$\Delta(\varphi^2) = 2(\lambda - v(x))\varphi^2 - 2|\nabla\varphi|^2 \quad (107)$$

adding $2(b - \lambda)\varphi^2$ on both sides of this equality, we obtain

$$[-\Delta + 2(b - \lambda)]\varphi^2 = -2(v(x) - b)\varphi^2 - 2|\nabla\varphi|^2 \quad (108)$$

Choosing $\lambda < b < E$ the right hand side of (108) is nonpositive for $|x|$ large enough.

Now set

$$u(x) = \varphi^2(x) - M\mathcal{E}(x) \quad (109)$$

where $\mathcal{E}(x)$ is the fundamental solution of the operator $-\Delta + k^2$ with

$$k = \sqrt{2(b - \lambda)} \quad (110)$$

Choose R so large that $\mathcal{E}(x) > 0$ and $v(x) > b$ for $|x| > R$. Now choose M so large that $u(x) < 0$ on $\{x \in \bar{\Omega} : |x| = R\}$. We shall prove that

$$u(x) \leq 0 \quad (111)$$

on $\{x \in \bar{\Omega} : |x| = R\}$ from which the theorem will follow. Subtracting from (108) the equation

$$[-\Delta + 2(b - \lambda)] M\mathcal{E}(x) = 0, \quad (112)$$

we find that (103) is satisfied for $u(x)$ with

$$f = -2(v(x) - b)\varphi^2 - 2|\nabla\varphi|^2, \quad \text{for } |x| \geq R. \quad (113)$$

We then apply the maximum principle in each connected component of the subset

$$\Omega_{R,\rho} = \{x \in \bar{\Omega} : R \leq |x| \leq \rho\} \quad (114)$$

to the function

$$u^\varepsilon(x) = \int u(x-y)\eta_\varepsilon(y)dy \quad (115)$$

where $\eta_\varepsilon(x) = \varepsilon^{-m}\eta(\frac{x}{\varepsilon})$ and $\eta(x)$ is the mollifier. Recall that $\eta(x)$ is given by

$$\eta(x) = \begin{cases} e^{-\left(\frac{1}{1-|x|^2}\right)} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We indeed have

$$(-\Delta + k^2)u^\varepsilon = f^\varepsilon = \int f(x-y)\eta_\varepsilon(y)dy \leq 0 \quad (116)$$

$u \in L^1(\mathbb{R}^\Lambda)$ implies that $u^\varepsilon(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Set

$$M_\rho(\varepsilon) = \max_{\{x \in \bar{\Omega} : |x| = \rho\}} |u^\varepsilon(x)| \quad (117)$$

since $u(x) < 0$ for $x \in \{x \in \bar{\Omega} : |x| = R\}$, using the fact that $u^\varepsilon(x) \rightrightarrows u(x)$ as $\varepsilon \rightarrow 0$ on $\{x \in \bar{\Omega} : |x| = R\}$, we conclude that $u^\varepsilon(x) < 0$ on $\{x \in \bar{\Omega} : |x| = R\}$ for small ε .

It then follows from lemma 17 that

$$u^\varepsilon(x) \leq M_\rho(\varepsilon) \quad \text{for } x \in \Omega_{R,\rho}. \quad (118)$$

Letting $\rho \rightarrow \infty$, we get

$$u^\varepsilon(x) \leq 0 \quad \text{for } x \in \bar{\Omega} \text{ and } |x| \geq R. \quad (119)$$

Now since

$$u^\varepsilon(x) \rightrightarrows u(x) \text{ as } \varepsilon \rightarrow 0 \quad (120)$$

in every relatively compact subset of $\{x \in \bar{\Omega} : |x| \geq R\}$, it follows that

$$u(x) \leq 0 \quad \text{for } x \in \{x \in \bar{\Omega} : |x| \geq R\} \quad (121)$$

■

Corollary 18 *If $v(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, then for any eigenfunction φ of the boundary value problem in theorem 7 satisfies, the following estimate*

$$|\varphi(x)| \leq C_a e^{-a|x|} \quad (122)$$

where $a > 0$ is arbitrary and $C_a > 0$.

Theorem 19 (Helffer-Sjostrand [8]) *The L^2 -solution u of*

$$\begin{cases} \left(-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1\right) u = 0 & \text{in } B^c \\ u = \varphi & \text{on } \partial B \text{ (in the trace sense)} \end{cases} \quad (E) \quad (123)$$

satisfies

$$u(x) = e^{-\frac{x^2}{4}} |x|^{-1/2} h(x) \quad (124)$$

where

$$\partial^\beta h(x) = O(|x|^{-|\beta|}) \quad \forall \beta \in \mathbb{N}^m \quad (125)$$

Using the change of variable $v = e^{\Phi/2} u$ and applying this theorem to each component of u , we obtain

Corollary 20 *The L^2 -solution v of the system*

$$(-\Delta + \nabla \Phi \cdot \nabla) \mathbf{v} + \mathbf{Hess} \Phi \mathbf{v} = \nabla g \quad \text{in } \mathbb{R}^\Lambda \quad (126)$$

satisfies

$$\lim_{|x| \rightarrow \infty} \partial^\alpha \mathbf{v}(x) = 0 \quad \forall \alpha \in \mathbb{N}^m \quad (127)$$

Proof of theorem 8 (Sjöstrand [14]). Denote by

$$K : C^\infty(\partial B) \rightarrow C^\infty(B^c) \quad (128)$$

the operator that assigns each boundary value the corresponding solution. Since by theorem 16

$$\lim_{|x| \rightarrow \infty} u(x) = 0, \quad (129)$$

the maximum principle implies that K is monotone increasing. Indeed, $Kg \geq 0$ whenever $g \geq 0$. This implies that the operator K is increasing and that $Kg \leq \sup g$, if $\sup g \geq 0$, $Kg \geq \inf g$ if $\inf g \leq 0$.

Let

$$u_o = K1 \quad (\geq 0) \quad (130)$$

which is a radial function i.e.

$$u_o(x) = u_o(|x|); \quad (131)$$

with

$$\left[-\partial_r^2 - \left(\frac{m-1}{r}\right) \partial_r + \frac{r^2}{4} - \frac{m}{2} + 1\right] u_o(r) = 0, \quad u_o(R) = 1. \quad (132)$$

We perform the Liouville's transformation

$$u_o = r^{-(m-1)/2} f(r) \quad (133)$$

to get rid of the term involving ∂_r . We finally get

$$\left[-\partial_r^2 + \frac{r^2}{4} - \frac{(m-1)(m-3)}{4r^2} + 1 - \frac{m}{2} \right] f(r) = 0, \quad f(R) = R^{(m-1)/2}. \quad (134)$$

which we write in the form

$$[-\partial_r^2 + V(r)] f(r) = \frac{m}{2} f(r), \quad f(R) = R^{(m-1)/2}. \quad (135)$$

where

$$V(r) = \frac{r^2}{4} - \frac{(m-1)(m-3)}{4r^2} + 1 \rightarrow \infty \text{ as } r \rightarrow \infty. \quad (136)$$

Since

$$\int_{r_o}^{\infty} \frac{|V'(r)|^2}{|V(r)|^{5/2}} dr < \infty \quad \text{and} \quad \int_{r_o}^{\infty} \frac{|V''(r)|^2}{|V(r)|^{3/2}} dr < \infty \quad \text{for some large } r_o, \quad (137)$$

Classical results on Schrodinger operators (see appendix) allow us to have the asymptotics of $f(r)$ as following:

$$f_{\pm}(r) = Cr^{-1/2} e^{\pm \frac{r^2}{4}} (1 + o(1)). \quad (138)$$

Now since $u_o \rightarrow 0$ as $r \rightarrow \infty$, we conclude that

$$f(r) = f_-(r) = Cr^{-1/2} e^{-\frac{r^2}{4}} (1 + o(1)) \quad (139)$$

Hence

$$u_o(r) = Cr^{-\frac{m}{2}} e^{-\frac{r^2}{4}} (1 + o(1)) > 0 \quad (140)$$

Next, we write

$$u(x) = j(x) u_o(r) \quad (141)$$

Let $g \in C^\infty(\partial B)$ be strictly positive everywhere and let

$$u = Kg. \quad (142)$$

Denote by $g_{\min} = \inf g$ and $g_{\max} = \sup g$. We obviously have

$$g_{\min} u_o \leq u \leq g_{\max} u_o. \quad (143)$$

Hence,

$$j(x) = \frac{u(x)}{u_o(x)} \quad (144)$$

is bounded.

Next, we perform a change in polar coordinates (r, θ) by setting $x = r\theta$. Under this change of coordinates, the operator

$$-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1 \quad (145)$$

becomes

$$-\partial_r^2 - \left(\frac{m-1}{r}\right) \partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2} \Delta_\theta \quad (146)$$

where Δ_θ is the Laplace-Beltrami operator on S^{m-1} .

Since the operator $-\Delta + \frac{x^2}{4} - \frac{m}{2} + 1$ is rotationally invariant and $\partial_\theta^\alpha u$ takes continuously the value $\partial_\theta^\alpha g$ on ∂B , using the fact that each $\partial_\theta^\alpha u$ arises as infinitesimal rotation, we conclude that for every α , $\partial_\theta^\alpha u$ is a solution of the boundary value problem (E) (under the change of coordinates) with

$$\partial_\theta^\alpha u = \partial_\theta^\alpha g \quad \text{on } \partial B. \quad (147)$$

Therefore,

$$\partial_\theta^\alpha u = O(1)e^{-\frac{r^2}{4}}, \quad \forall \alpha \in \mathbb{N}^m, \quad (148)$$

which implies

$$\partial_\theta^\alpha j = O(1), \quad \forall \alpha \in \mathbb{N}^m. \quad (149)$$

Now we need to control some radial derivative of j .

In polar coordinates, we have

$$\left[-\partial_r^2 - \left(\frac{m-1}{r}\right) \partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2} \Delta_\theta \right] u_o(r) = 0 \quad (150)$$

Write

$$\left[-\partial_r^2 - \left(\frac{m-1}{r}\right) \partial_r + \frac{r^2}{4} - \frac{m}{2} + 1 - r^{-2} \Delta_\theta \right] j(r, \theta) u_o(r) = 0. \quad (151)$$

Using (132) and the product rule of differentiation, (151) becomes

$$\left[\partial_r^2 + \left[2 \frac{\partial_r u_o}{u_o} + \left(\frac{m-1}{r}\right) \right] \partial_r \right] j = -r^{-2} \Delta_\theta j \quad (152)$$

Here

$$\partial_\theta^\alpha (r^{-2} \Delta_\theta j) = O(r^{-2}), \quad \forall \alpha \in \mathbb{N}^m, \quad (153)$$

and

$$\frac{\partial_r u_o}{u_o} = -\frac{r}{2} + O\left(\frac{1}{r}\right). \quad (154)$$

Thus, (152) can be written as

$$\left[\partial_r^2 + \left[-r + O\left(\frac{1}{r}\right) \right] \partial_r \right] j = O(r^{-2}) \quad (155)$$

Let

$$\varphi(r) = r + O\left(\frac{1}{r}\right). \quad (156)$$

We have

$$[\partial_r - f(r)] \partial_r j = O(r^{-2}) \quad (157)$$

Let

$$F(r) = \int_1^r f(t) dt \sim r^2. \quad (158)$$

Solving (157), we get

$$\partial_r j = - \int_r^\infty e^{F(r)-F(s)} [O(s^{-2})] ds + C e^{F(r)}. \quad (159)$$

Since

$$F(r) - F(s) \sim r^2 - s^2 \leq 2r(r-s) \text{ for } s \geq r, \quad (160)$$

$\partial_r j$ cannot tend to $\pm\infty$ when $r \rightarrow \infty$, we conclude that $C = 0$ and

$$\partial_r j = - \int_r^\infty e^{F(r)-F(s)} [O(s^{-2})] ds = O(r^{-3}) \quad (161)$$

More generally, since $\partial_\theta^\alpha j$ is a solution of (157) with right hand side

$$-r^{-2} \partial_\theta^\alpha (\Delta_\theta j) = O(r^{-2}), \quad (162)$$

using the same argument as above with j replaced by $\partial_\theta^\alpha j$, we have

$$\partial_r \partial_\theta^\alpha j = O(r^{-3}) \quad (163)$$

Now differentiating

$$[\partial_r - f(r)] \partial_r \partial_\theta^\alpha j = O(r^{-2}) \quad (164)$$

with respect to r , we get

$$[\partial_r - f(r)] \partial_r^2 \partial_\theta^\alpha j = O(r^{-3}), \quad (165)$$

using again the same argument as before, we get

$$\partial_r^2 \partial_\theta^\alpha j = O(r^{-4}) \quad (166)$$

continuing this way, we finally get

$$\partial_r^k \partial_\theta^\alpha j = O(r^{-2-k}) \quad k = 1, 2, \dots \quad (167)$$

Going back to x -coordinates, we get

$$\partial^\alpha j(x) = O(|x|^{-|\alpha|}), \quad \forall \alpha \in \mathbb{N}^m, \alpha \neq 0. \quad (168)$$

■

7 Weighted Estimates for the Decay of Correlation

tion

In this section, we propose to get estimates suitable for obtaining the decay of the correlation functions. We shall first analyze the case where Ψ and the source term g are compactly supported

7.1 The compactly supported case.

We shall assume that Φ is given by

$$\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^\Lambda. \quad (169)$$

where

$$|\partial^\alpha \nabla \Psi| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \quad (170)$$

Again g will denote a smooth function on \mathbb{R}^Γ with lattice support $S_g = \Gamma$. We shall identify g with \tilde{g} defined on \mathbb{R}^Λ and shall assume that

$$|D^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{|\Gamma|} \quad (171)$$

In addition, we shall momentarily assume that Ψ is compactly supported in \mathbb{R}^Λ and g is compactly supported in \mathbb{R}^Γ but these assumptions will be relaxed later on.

Let M be the diagonal matrix

$$M = (\delta_{ij} \rho(i))_{i,j \in \Lambda}$$

where ρ is a weight function on Λ satisfying

$$e^{-\lambda} \leq \frac{\rho(i)}{\rho(j)} \leq e^\lambda, \quad \text{if } i \sim j \text{ for some } \lambda > 0 \quad (172)$$

Assume also that there exists $\delta_o \in (0, 1)$ such that

$$M^{-1} \mathbf{Hess} \Phi(x) M \geq \delta_o \quad (173)$$

for every M as above.

Let

$$\rho(i) = e^{\kappa d(i, S_g)} \quad (174)$$

where κ is a positive. Define

$$|x|_{2,\rho} := \left(\sum_{i \in \Lambda} \rho(i)^2 x_i^2 \right)^{1/2}$$

Let f be the solution of the equation

$$\begin{cases} -\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle \\ \langle f \rangle_{L^2(\mu)} = 0 \end{cases}$$

Recall that ∇f is a solution of the system

$$(-\Delta + \nabla \Phi \cdot \nabla) \nabla f + \mathbf{Hess} \Phi \nabla f = \nabla g \quad \text{in } \mathbb{R}^\Lambda \quad (175)$$

Let $t_1 = (t_i)_i \in \mathbb{R}^\Lambda$

$$\langle \nabla (\nabla \Phi \cdot \nabla f), t_1 \rangle = \sum_{i,k \in \Lambda} (f_{x_i} \Phi_{x_i x_k} t_k + \Phi_{x_i} f_{x_i x_k} t_k) \quad (176)$$

$$= \langle \nabla f, \mathbf{Hess} \Phi t_1 \rangle + \nabla \Phi \cdot \nabla \langle \nabla f, t_1 \rangle \quad (177)$$

On the other hand,

$$\langle \nabla (\Delta f), t_1 \rangle = \Delta \langle \nabla f, t_1 \rangle.$$

We therefore have

$$\langle \nabla g, t_1 \rangle = (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla f, t_1 \rangle + \langle \nabla f, \mathbf{Hess} \Phi t_1 \rangle \quad (178)$$

Because $\nabla f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we consider a point x_o at which

$$|\nabla f(x)|_{2,\rho} = \left(\sum_{i \in \Lambda} \rho(i)^2 f_{x_i}^2(x) \right)^{1/2}$$

is maximal.

If M is the diagonal matrix

$$M = (\delta_{ij} \rho(i))$$

we have

$$\langle \nabla g, M t_1 \rangle = (\nabla \Phi \cdot \nabla - \Delta) \langle \nabla f, M t_1 \rangle + \langle \nabla f, \mathbf{Hess} \Phi M t_1 \rangle \quad (179)$$

Now choose

$$t_1 = (\rho(i) f_{x_i}(x_o))_{i \in \Lambda}.$$

We need the following lemma.

Lemma 21 *Under the assumptions and notations above, the function*

$$x \mapsto \langle \nabla f(x), M t_1 \rangle$$

achieves its maximum value at x_o

Proof. Let

$$\zeta(x) = \langle \nabla f(x), Mt_1 \rangle \quad (180)$$

and

$$\pi(x) = |\nabla f(x)|_{2,\rho}. \quad (181)$$

Again by the maximum principle, the function $\zeta(x)$ achieves its maximum at some $\bar{x}_o \in \mathbb{R}^\Lambda$. It is easy to see that x_o is a critical point for $\zeta(x)$. Moreover, for any $a \in \mathbb{R}^\Lambda$, we have

$$\langle a, \mathbf{Hess}\pi(x_o)a \rangle \quad (182)$$

$$= 2 \langle a, \mathbf{Hess}\zeta(x_o)a \rangle + 2 \sum_{j,k} \left(\sum_i f_{x_i x_j}(x_o) f_{x_i x_k}(x_o) \rho(i)^2 \right) a_j a_k \quad (183)$$

$$= 2 \langle a, \mathbf{Hess}\zeta(x_o)a \rangle + 2 \left\| \sum_j M(a_j \nabla f_{x_j})(x_o) \right\|^2. \quad (184)$$

Because $\langle a, \mathbf{Hess}\pi(x_o)a \rangle < 0$, we must have $\langle a, \mathbf{Hess}\zeta(x_o)a \rangle < 0$ for any $a \in \mathbb{R}^\Lambda$. Thus, x_o is a local maximum for $\zeta(x)$. Moreover, on one hand, we have

$$\zeta(\bar{x}_o) \geq \zeta(x_o) = \pi(x_o). \quad (185)$$

On the other hand, Cauchy-Schwartz gives

$$\zeta(\bar{x}_o) \leq [\pi(\bar{x}_o)]^{1/2} [\pi(x_o)]^{1/2} \quad (186)$$

$$\leq \pi(x_o) \quad (187)$$

These last two above inequalities imply

$$\zeta(\bar{x}_o) = \zeta(x_o) \quad (188)$$

and the result follows. ■

Now using lemma 21 above, we have

$$(\nabla\Phi \cdot \nabla - \Delta) \langle \nabla f(x_o), Mt_1 \rangle \geq 0.$$

This, then implies

$$\begin{aligned} \langle \nabla g(x_o), Mt_1 \rangle &\geq \langle \nabla f(x_o), \mathbf{Hess}\Phi(x_o)Mt_1 \rangle \\ &= \langle M\nabla f(x_o), M^{-1}\mathbf{Hess}\Phi(x_o)Mt_1 \rangle \\ &= \langle t_1, M^{-1}\mathbf{Hess}\Phi(x_o)Mt_1 \rangle \\ &\geq \delta_o |\nabla f(x_o)|_{2,\rho} \end{aligned}$$

Thus

$$\begin{aligned} |\nabla f(x_o)|_{2,\rho}^2 &\leq \frac{1}{\delta_o} \langle M\nabla g(x_o), t_1 \rangle \\ &= \frac{1}{\delta_o} \|M\nabla g(x_o)\| |\nabla f(x_o)|_{2,\rho} \end{aligned}$$

If

$$|\nabla f(x_o)|_{2,\rho} = 0 \quad (189)$$

there is nothing to prove otherwise we have

$$\begin{aligned} \left(\sum_{i \in \Lambda} f_{x_i}^2(x_o) \rho^2(i) \right)^{1/2} &\leq \frac{1}{\delta_o} \left(\sum_{i \in \Lambda} g_{x_i}^2(x_o) \rho^2(i) \right)^{1/2} \\ &= \frac{1}{\delta_o} \left(\sum_{i \in S_g} g_{x_i}^2(x_o) e^{2\kappa d(i, S_g)} \right)^{1/2} \\ &\leq \frac{1}{\delta_o} \left(\sum_{i \in S_g} g_{x_i}^2(x_o) \right)^{1/2} \end{aligned}$$

We have proved:

Proposition 22 *Let g be a smooth function satisfying*

$$|D^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{|\Gamma|} \quad (190)$$

and Φ is as above. If f is the unique C^∞ -solution of the equation

$$\begin{cases} -\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle \\ \langle f \rangle_{L^2(\mu)} = 0, \end{cases}$$

then

$$\sum_{i \in \Lambda} f_{x_i}^2(x) e^{2\kappa d(i, S_g)} \leq C \quad \forall x \in \mathbb{R}^\Lambda$$

C and κ are positive constants that could possibly depend on the size of the support of g but do not depend on Λ and f

Corollary 23 *Let g and h be smooth functions on \mathbb{R}^Γ , and $\mathbb{R}^{\Gamma'}$ where Γ and $\Gamma' \not\subseteq \Lambda$ with $\Gamma \cap \Gamma' = \emptyset$ denote respectively the support of g and h and assume that g and h satisfy (4.22). Then under the assumptions of proposition 2, we have*

$$|\mathbf{cov}(g, h)| \leq C e^{-\kappa d(S_h, S_g)} \quad (191)$$

where C and κ are positive constants that do not depend on Λ , but possibly dependent on the size of the supports of g and h .

Proof. Using the formula for the representation of the covariance, we have

$$\begin{aligned}
|\mathbf{cov}(g, h)| &= \left| \left\langle A_{\Phi}^{1^{-1}} \nabla g \cdot \nabla h \right\rangle \right| \\
&= |\langle \nabla f \cdot \nabla h \rangle| \\
&= \int \sum_{i \in \Lambda} f_{x_i}(x) e^{\kappa d(i, S_g)} e^{-\kappa d(i, S_g)} h_{x_i} d\mu(x) \\
&\leq \int \left(\sum_{i \in \Lambda} f_{x_i}^2(x) e^{2\kappa d(i, S_g)} \right)^{1/2} \left(\sum_{i \in S_h} h_{x_i}^2(x) e^{-2\kappa d(i, S_g)} \right)^{1/2} d\mu(x) \\
&\leq \left[\int \sum_{i \in \Lambda} f_{x_i}^2(x) e^{2\kappa d(i, S_g)} d\mu(x) \right]^{1/2} \left[\int \sum_{i \in S_h} h_{x_i}^2(x) e^{-2\kappa d(i, S_g)} d\mu(x) \right]^{1/2} \\
&\leq C \left(\sum_{i \in S_g} g_{x_i}^2(x_o) \right)^{1/2} \left[\int \sum_{i \in S_h} h_{x_i}^2(x) d\mu(x) \right]^{1/2} e^{-\kappa d(S_h, S_g)}
\end{aligned}$$

■

Remark 24 *This is the higher dimensional version of theorem 1.4 in [8]. Notice that our proof does not require the assumptions (1.17) and (1.19) namely*

$$\|\mathbf{Hess}\Phi(x)\|_{\mathcal{L}(l_{\rho}^{\infty})} \leq C$$

and

$$\|\mathbf{Hess}\Psi(x)\|_{\mathcal{L}(l_{\rho}^{\infty})} \leq \rho < 1$$

for all ρ as above. However, we required that Φ satisfies

$$M^{-1} \mathbf{Hess}\Phi(x) M \geq \delta_o$$

for some $\delta_o \in (0, 1)$ and M as above.

7.2 Relaxing the Compact Support Assumptions.

We propose now to relax the assumptions of compact support made previously on Ψ and g .

As before, let M be the diagonal matrix

$$M = (\delta_{ij} \rho(i))$$

where ρ is given by

$$\rho(i) = e^{\kappa d(i, S_g)} \tag{192}$$

and

$$M^{-1} \mathbf{Hess}\Phi(x) M \geq \delta_o \tag{193}$$

for every M as above.

Next, we propose to generalize the results in propositions 22 without the assumptions of compact support on Ψ and g by means of a family of cutoff functions. Let us introduce as in [8] a family cutoff functions

$$\chi = \chi_\varepsilon \tag{194}$$

($\varepsilon \in [0, 1]$) in $\mathcal{C}_o^\infty(\mathbb{R})$ with value in $[0, 1]$ such that

$$\begin{cases} \chi = 1 & \text{for } |t| \leq \varepsilon^{-1} \\ |\chi^{(k)}(t)| \leq C_k \frac{\varepsilon}{|t|^k} & \text{for } k \in \mathbb{N} \end{cases}$$

We could take for instance

$$\chi_\varepsilon(t) = f(\varepsilon \ln |t|)$$

for a suitable f .

We then introduce

$$\Psi_\varepsilon(x) = \chi_\varepsilon(|x|)\Psi, \quad x \in \mathbb{R}^\Lambda \tag{195}$$

and

$$g_\varepsilon(x) = \chi_\varepsilon(|x|)g \quad x \in \mathbb{R}^\Gamma \tag{196}$$

Recall that

$$-\Delta f + \nabla\Phi \cdot \nabla f = g - \langle g \rangle_{t,\Lambda} \tag{197}$$

which implies

$$(-\Delta + \nabla\Phi \cdot \nabla) \otimes \mathbf{v} + \mathbf{Hess}\Phi \mathbf{v} = \nabla g \tag{198}$$

where

$$\mathbf{v} = \nabla f$$

Under the transformations

$$\mathbf{v} = e^{-\Phi/2} \mathbf{u} \quad \text{and} \quad \mathbf{q} = e^{-\Phi/2} \nabla g$$

we have

$$\left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} \right) \otimes \mathbf{Iu} + \mathbf{Hess}\Phi \mathbf{u} = \mathbf{q} \quad \text{in } \mathbb{R}^\Lambda \tag{199}$$

We first verify that the assumptions on Ψ and g are satisfied by $\Psi_\varepsilon(x)$ and $g_\varepsilon(x)$. Namely

$$|\partial^\alpha \nabla \Psi| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{200}$$

$$|\partial^\alpha \nabla g| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{201}$$

and

$$M^{-1} \mathbf{Hess}\Phi M \geq \delta > 0, \quad 0 < \delta < 1 \tag{202}$$

M shall still denote the diagonal matrix

$$M = (\delta_{ij} \rho(i))_{i,j \in \Lambda}$$

where ρ is a weight function on \mathbb{R}^Λ satisfying

$$e^{-\lambda} \leq \frac{\rho(i)}{\rho(j)} \leq e^\lambda, \quad \text{if } i \sim j \text{ for some } \lambda > 0 \quad (203)$$

Using

$$\mathbf{Hess}\Psi \geq \delta - 1,$$

we obtain immediately

$$M^{-1}\mathbf{Hess}\Psi_\varepsilon(x)M \geq (\delta - 1)\chi_\varepsilon(|x|) - C\varepsilon \quad (204)$$

for all ε and some constant C . Indeed, we know that

$$M^{-1}\mathbf{Hess}\Psi(x)M \geq (\delta - 1)$$

for simplicity, we shall write

$$\chi_\varepsilon = \chi \quad \text{and } r = |x|$$

$$\Psi_\varepsilon(x) = \chi(r)\Psi(x)$$

$$\begin{aligned} \frac{\rho(j)}{\rho(i)}\Psi_{\varepsilon x_i x_j} &= \frac{1}{r} \frac{\rho(j)}{\rho(i)} \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \chi'(r)\Psi + \frac{\rho(j)}{\rho(i)} \frac{x_i x_j}{r^2} \chi''(r)\Psi \\ &\quad + \frac{\rho(j)}{\rho(i)} \frac{x_j}{r} \chi'(r)\Psi_{x_j} + \frac{\rho(j)}{\rho(i)} \chi(r)\Psi_{x_i x_j} \end{aligned}$$

Let $a \in \mathbb{R}^\Lambda$,

$$\begin{aligned} &\langle M^{-1}\mathbf{Hess}\Psi_\varepsilon(x)Ma, a \rangle \\ &= \left(\frac{1}{r} \sum_i a_i^2 - \frac{1}{r^3} \sum_{i,j} \frac{\rho(j)}{\rho(i)} a_i a_j x_i x_j \right) \chi'(r)\Psi \\ &\quad + \frac{1}{r^2} \chi''(r)\Psi \sum_{i,j} \frac{\rho(j)}{\rho(i)} a_i a_j x_i x_j + \frac{1}{r} \chi'(r) \sum_{i,j} \frac{\rho(j)}{\rho(i)} a_i a_j x_j \Psi_{x_j} \\ &\quad + \chi(r) \sum_{i,j} \frac{\rho(j)}{\rho(i)} a_i a_j \Psi_{x_i x_j} \\ &\geq -2 \frac{a^2}{r} |\chi'(r)\Psi(x)| - a^2 |\chi''(r)\Psi(x)| - C |\chi'(r)| a^2 + (\delta - 1) \chi(r) a^2 \\ &\geq [(\delta - 1) \chi(r) - \varepsilon C] a^2 \end{aligned}$$

We conclude that

$$M^{-1}\mathbf{Hess}\Psi_\varepsilon(x)M \geq (\delta - 1)\chi(r) - \varepsilon C$$

for all $\varepsilon > 0$

It follows that

$$M^{-1}\mathbf{Hess}\Phi_\varepsilon(x)M \geq \delta - C\varepsilon. \quad (205)$$

Now with δ replaced by $\delta' = \delta - C\varepsilon$, we see that

$$M^{-1}\mathbf{Hess}\Phi_\varepsilon(x)M \geq \delta', \quad 0 < \delta' < 1 \quad (206)$$

for ε small enough. (Notice that ε is possibly Λ -depend)It remains to check the assumptions on g_ε and Ψ_ε .

To see that

$$|\partial^\alpha \nabla g_\varepsilon| \leq C + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}, \quad (207)$$

we have

$$g_\varepsilon(x) = \chi_\varepsilon(r)g(x), \quad x \in \mathbb{R}^\Gamma.$$

Again let $|\alpha| \geq 1$, using Leibniz's formula, we have

$$\begin{aligned} |\partial^\alpha g_\varepsilon| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g \\ &= |\partial^\alpha g| + |g \partial^\alpha \chi_\varepsilon(r)| + \sum_{\substack{\beta < \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} |\partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g| \end{aligned}$$

With the assumption $g(0) = 0$, we write

$$|g(x)| \leq \int_0^1 \sum_{j \in \Lambda} |x_j g_{x_j}(sx)| ds \quad (208)$$

$$\leq \int_0^1 \left(\sum_{j \in \Lambda} x_j^2 \right)^{1/2} \left(\sum_{j \in \Lambda} g_{x_j}^2(sx) \right)^{1/2} ds \quad (209)$$

$$\leq C_g r \quad (210)$$

again using the fact that

$$r \partial^\alpha \chi_\varepsilon(r) = \mathcal{O}_\alpha(\varepsilon),$$

we get

$$|g \partial^\alpha \chi_\varepsilon(r)| = \mathcal{O}_{\alpha,\Lambda}(\varepsilon). \quad (211)$$

observe also that

$$\sum_{\substack{\beta < \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} |\partial^\beta \chi_\varepsilon(r) \partial^{\alpha-\beta} g| = \mathcal{O}_\alpha(\varepsilon) \quad (212)$$

it then immediately follows from the assumption on g that

$$|\partial^\alpha \nabla g_\varepsilon| \leq C_\alpha + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Gamma|}, \quad (213)$$

Similarly, one can prove that

$$|\partial^\alpha \nabla \Psi_\varepsilon| \leq C_\alpha + \mathcal{O}_{\alpha,\Lambda}(\varepsilon), \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \quad (214)$$

Thus Ψ_ε and g_ε are compactly supported and satisfy all the conditions that were previously required on Ψ and g .

If \mathbf{u}_ε denotes the family of solutions corresponding to the family of datas Φ_ε and g_ε , one can see that \mathbf{u}_ε converges to \mathbf{u} in C^∞ . The proof which based on regularity estimates is given in detail in [8],[14]. As a consequence, the family of solution $\mathbf{v}_\varepsilon = e^{\Phi_\varepsilon} \mathbf{u}_\varepsilon$ in the weighted spaces converges to \mathbf{v} in C^∞

Proposition 25 *If $g(0) = 0$, then Proposition 22 holds without the assumptions of compact support on Ψ and g .*

Proof. Using proposition 2 we have

$$\left(\sum_{i \in \Lambda} f_{\varepsilon x_i}^2(x) e^{2\kappa d(i, S_g)} \right)^{1/2} \leq C |S_g|^{1/2} + \mathcal{O}_\Lambda(\varepsilon) \quad \forall x \in \mathbb{R}^\Lambda.$$

The result follows by taking the limit as $\varepsilon \rightarrow 0$ ■

Corollary 26 *If $g = x_i$ and $h = x_j$ we get*

$$|\text{cor}(i, j)| \leq C e^{-\kappa d(i, j)}$$

Which shows that the corresponding model does not exhibit a phenomenon of phase transition.

Example 27 (The Kac-like model) *An example of a nonquadratic model satisfying the assumptions above is given by*

$$\Phi_\Lambda(x) = \frac{x^2}{2} - \sum_{i \sim j} \ln \cosh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]$$

The summation is over all nearest neighbor sites.

$$\Psi(x) = - \sum_{i, j \in \Lambda, i \sim j} \ln \cosh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]$$

with $\nu > 0$ small enough.

$$\Psi_{x_i} = - \sum_{j: j \sim i} \frac{\sqrt{\frac{\nu}{2}} \sinh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]}{\cosh \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]}$$

$$\Psi_{x_i x_k} = \begin{cases} -\frac{\nu}{2} \sum_{j: j \sim i} \frac{1}{\cosh^2 \left[\sqrt{\frac{\nu}{2}} (x_i + x_j) \right]} & \text{if } k = i \\ -\frac{\nu}{2} \frac{1}{\cosh^2 \left[\sqrt{\frac{\nu}{2}} (x_i + x_k) \right]} & \text{if } k \sim i \\ 0 & \text{otherwise} \end{cases}$$

It then follows that

$$|\Psi_{x_i}| \leq 2d\sqrt{\frac{\nu}{2}},$$

$$|\Psi_{x_i x_i}| \leq d\nu,$$

and

$$|\Psi_{x_i x_k}| \leq \frac{\nu}{2} \quad \text{if } k \sim i$$

Similarly, using the properties of \cosh and \sinh and the fact that $\sinh t \leq \cosh t$ for all t one can see that all derivative of order greater than or equal to one are bounded.

Now we propose to check that for ν small enough, the Kac Hamiltonian satisfies

$$M^{-1}\mathbf{Hess}\Phi(x)M \geq \delta_o$$

for some $\delta_o \in (0, 1)$ and M as above.

We need the following lemma.

Lemma 28 (Schur's Lemma- The R and C bound) *For each rectangular array*

$$(c_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

and each pair of sequence $(x_i)_{1 \leq i \leq m}$ and $(y_j)_{1 \leq j \leq n}$ we have the bound

$$\left| \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i y_j \right| \leq \sqrt{RC} \left(\sum_{i=1}^m |x_i|^2 \right)^{1/2} \left(\sum_{j=1}^n |y_j|^2 \right)^{1/2}$$

where R and C are the row sum and column sum maxima defined by

$$R = \max_i \sum_{j=1}^n |c_{ij}| \quad \text{and} \quad C = \max_j \sum_{i=1}^m |c_{ij}|$$

This bound is known as Schur's Lemma, but, ironically, it may be the second most famous result with this name. The schur's decomposition lemma for $n \times n$ matrices is also known under this name. Nevertheless, this inequality is surely the single most commonly used tool for estimating a quadratic form.

Going back to the example, we have for any $a = (a_i)_{i \in \Lambda} \in \mathbb{R}^\Lambda$,

$$\begin{aligned} & \langle M^{-1}\mathbf{Hess}\Phi Ma, a \rangle \\ &= \sum_{i,j} \Phi_{x_i x_j} \frac{\rho(i)}{\rho(j)} a_i a_j \\ &= \sum_i \Phi_{x_i x_i} a_i^2 + \sum_{i \sim j} \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} a_i a_j \\ &\geq (1 - d\nu) a^2 + \sum_{i \sim j} \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} a_i a_j \end{aligned}$$

Now using the Schur's lemma above, we have

$$\begin{aligned} \left| \sum_{i \sim j} \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} a_i a_j \right| &\leq \sum_{i, j} \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} a_i a_j \right| \\ &\leq \sqrt{RC} a^2 \end{aligned}$$

where

$$R = \max_i \sum_j \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right|$$

and

$$C = \max_j \sum_i \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right|.$$

To estimate R , observe that

$$\sum_j \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right| = |\Psi_{x_i x_i}| + \sum_{j: j \sim i} \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right|$$

Now using the fact that

$$e^{-\kappa} \leq \frac{\rho(i)}{\rho(j)} \leq e^{\kappa} \quad \text{if } i \sim j$$

we have

$$\sum_j \left| \Psi_{x_i x_j} \frac{\rho(i)}{\rho(j)} \right| \leq d\nu + d\nu e^{\kappa}$$

Hence

$$R \leq d\nu (1 + e^{\kappa})$$

Similarly, we have

$$C \leq d\nu (1 + e^{\kappa})$$

Thus,

$$\begin{aligned} \langle M^{-1} \mathbf{Hess} \Phi M a, a \rangle &\geq [(1 - d\nu) - d\nu (1 + e^{\kappa})] a^2 \\ &= 1 - d\nu (2 + e^{\kappa}). \end{aligned}$$

The result follows by choosing $0 < \kappa < \ln 2$ and $\nu < \frac{1}{d(2 + e^{\kappa})}$

A The One dimensional Schrodinger Operator

In this appendix, the reader is also referred to [65] for a detailed exposition about the asymptotic behavior of the solutions of the one dimensional Schrodinger equation.

We set

$$Hy = -y'' + vy \quad (215)$$

where $v = v(x)$ is measurable, real-valued and locally bounded. Our objective in this section is to find the asymptotic behavior as $x \rightarrow \infty$ of eigenfunctions of the operator H whose potential $v(x)$ either grows or does not decrease rapidly enough as $x \rightarrow \infty$. We confine ourselves to examining the equation

$$Hy = k^2y \quad (216)$$

with real value of k^2 (k itself can be real or purely imaginary). First, we investigate the case of an integrable potential and then obtain some results for the case of a nonintegrable potential by means of the Liouville transformation.

If $v(x) = 0$ for $x \geq N$ then for $k \neq 0$, any solution of the equation (215) for $x \geq N$ turns into a linear combination of the functions e^{ikx} and e^{-ikx} . It is natural to expect that if $v(x)$ is decaying sufficiently rapidly, then the equation (215) has for $k \neq 0$ a fundamental system of two solutions, behaving as e^{ikx} and e^{-ikx} .

Theorem 29 *Let the potential $v(x)$ satisfy*

$$\int_{-\infty}^{\infty} |v(x)| dx < \infty \quad (217)$$

and let $k \neq 0$ be fixed. Then equation (215) has solutions $y_1(x)$ and $y_2(x)$, such that

$$y_1(x) = e^{ikx}(1 + o(1)) \quad (218)$$

$$y_2(x) = e^{-ikx}(1 + o(1)) \quad (219)$$

as $x \rightarrow \infty$.

Now let $k = \alpha + i\beta$, where $\beta \geq 0$. then the solution $y_1(x)$ satisfying condition (217) is uniquely defined and satisfies the integral equation

$$y(x) = e^{ikx} + \int_x^{\infty} \frac{\sin k(\xi - x)}{k} v(\xi) y(\xi) d\xi \quad (220)$$

Next, we propose to find the asymptotic behavior as $x \rightarrow \infty$ of eigenfunctions of H whose potential either grows or does not decrease rapidly enough as $x \rightarrow \infty$. Consider the general equation

$$y'' + p(x)y' + q(x)y = 0 \quad (221)$$

The following transformation will turn this equation into a simpler equation without the first derivative term.

A. Replacing the unknown function $y(x)$ by a function $z(x)$:

$$y(x) = a(x)z(x)$$

where $a(x)$ is a certain known function.

B. Change of the independent variable $t = \phi(x)$.

As a result of these transformations, a coefficient of the highest derivative may appear in equation (220), but we assume that the equation can be divided by it without loss of generality. Combining the transformation A and B we are able to reduce the broad class of equation (215) to an equation with an integrable potential.

First of all we note that A reduces the equation (220) to

$$z'' + \left(\frac{2a'}{a} + p \right) z' + \left(\frac{a''}{a} + \frac{pa'}{a} + q \right) z = 0 \quad (222)$$

whence we see that we may get rid of z' by solving the equation $\frac{2a'}{a} + p = 0$ to get

$$a(x) = e^{-\frac{1}{2} \int_{x_0}^x p(s) ds} \quad (223)$$

From (221), we see that if $p(x)$ decays as $x \rightarrow \infty$, then $q(x)$ changes by a function decreasing as $p^2(x)$ (provided $p'(x)$ decays faster than $p(x)$).

Investigating the possibilities of B, consider the equation

$$y'' \pm f^2(x)y = 0 \quad (224)$$

where $f(x) > 0$. It is easy to see that after the change of independent variable, $t = \phi(x)$, the equation (223) turns into

$$\ddot{y} + \dot{y} \pm \frac{f^2(x)}{\phi'^2(x)} y = 0 \quad (225)$$

where \ddot{y}, \dot{y} are the derivatives with respect to t . It is natural to set in (224)

$$\phi(x) = \int_{x_0}^x f(s) ds \quad (226)$$

so that the coefficient of y becomes ± 1 . The equation (6.11) takes the form

$$\ddot{y} + \dot{y} \frac{f'}{f^2} \pm y = 0 \quad (227)$$

and if $\frac{f'}{f^2}$ decays, by substitution A we may expect to reduce (226) to an equation of the form (215) with an integrable potential. Note that (225) makes sense only if the relation

$$\int_{x_0}^{\infty} f(s) ds = \infty \quad (228)$$

holds, which we assume henceforth (this is true in practice for all known cases). The composite of the two transformations of (223) (first B then A), is called

Liouville's transformation. Formula (225) determines the change of independent variable. In order to change the unknown function, note that

$$e^{-\frac{1}{2} \int_{t_0}^t \frac{f'(x(t))}{f^2(x(t))} dt} = \frac{C}{\sqrt{f(x(t))}} \quad (229)$$

which can be verified by changing variable in the integral (C is a constant). Hence the desired change takes the form

$$y(x(t)) = \frac{z(t)}{\sqrt{f(x(t))}} \quad (230)$$

It is now easy to calculate the equation so obtained by Liouville's transformation. we formulate the result in the following form.

Proposition 30 *By changing the independent variable and the unknown function according to the formulas*

$$t = \int_{x_0}^x f(s) ds \quad (231)$$

$$z(t) = \sqrt{f(x(t))} y(s(t)) \quad (232)$$

equation $y'' \pm f^2(x)y = 0$ reduces to to the form

$$-\ddot{z} + \left(\frac{f''}{2f^3} - \frac{3f'}{4f^4} \right) z = \pm z \quad (233)$$

where f' , f'' are derivative with respect to x and \ddot{z} is the derivative with respect to t and f , f' , f'' are to be understood as $f(x(t))$, $f'(x(t))$, $f''(x(t))$.

Set

$$q(x) = \frac{f''}{2f^3} - \frac{3f'}{4f^4} \quad (234)$$

The form of equation (232) is that of (215) with integrable potential if and only if

$$\int_{t_0}^{\infty} |q(x(t))| dt < \infty, \quad (235)$$

or what is the same,

$$\int_{x_0}^{\infty} |q(x)| f(x) dx < \infty, \quad (236)$$

that is

$$\int_{x_0}^{\infty} \left| \frac{f''}{2f^2} - \frac{3f'}{4f^3} \right| dx < \infty, \quad (237)$$

Theorem 31 *If $f(x)$ satisfies (227) and (236) then the equation (223) has pair of solution with asymptotics*

$$\frac{1}{\sqrt{f(x)}} e^{\pm i \int_{x_0}^x f(s) ds} (1 + o(1)) \quad (238)$$

for the plus sign in the equation, and

$$\frac{1}{\sqrt{f(x)}} e^{\pm \int_{x_0}^x f(s) ds} (1 + o(1)) \quad (239)$$

for the minus sign.

This result follows from Theorem 29 with Liouville's transformation. Returning now to the equation (215) with non-integrable potential v , this equation clearly takes the form of (223) if we set

$$f(x) = \sqrt{\pm (k^2 - v(x))}. \quad (240)$$

We will consider separately three different mode of behavior of $v(x)$:

a) The case of increasing positive $v(x)$, that is

$$v(x) \rightarrow \infty \text{ as } x \rightarrow \infty;$$

b) The case of decaying $v(x)$, that is

$$v(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

c) the case of increasing negative $v(x)$, that is

$$v(x) \rightarrow -\infty \text{ as } x \rightarrow \infty$$

To apply theorem 31, we must impose the appropriate condition on $v(x)$. Note that condition (227) always automatically satisfied in cases a) and c) and in case b) for $k^2 \neq 0$ (we confine ourselves to this case). Condition (236) is satisfied if we impose on v the following restrictions:

Case a) and c):

$$\int_{x_0}^{\infty} \frac{|v'|}{|v|^{5/2}} dx < \infty, \quad \int_{x_0}^{\infty} \frac{|v''|}{|v|^{3/2}} dx \quad (241)$$

Case b)

$$\int_{x_0}^{\infty} |v'|^2 dx < \infty, \quad \int_{x_0}^{\infty} |v''|^2 dx \quad (242)$$

Clearly x_0 in the last four integrals above can be chosen arbitrarily large. We now reformulate theorem 31 in the most convenient form.

Theorem 32 *If in cases a)-c) the corresponding condition (240) or (241) are satisfied, then equation (215) has a pair of solutions with the following asymptotics as $x \rightarrow \infty$:*

in case a)

$$v(x)^{-1/4} e^{\pm \int_{x_0}^x \sqrt{v(x)-k^2} dx} (1 + o(1)) \quad (243)$$

in case b) ($k \neq 0$):

$$e^{\pm ik \int_{x_0}^x \sqrt{1 - \frac{v(x)}{k^2}} dx} (1 + o(1)) \quad (244)$$

in case c)

$$v(x)^{-1/4} e^{\pm i \int_{x_0}^x \sqrt{k^2 - v(x)} dx} (1 + o(1)) \quad (245)$$

The lower limit of integration x_0 can again be chosen arbitrarily large.

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