Report on the paper

Witten Laplacian Methods for the Decay of Correlations

by Assane Lo

In this paper the author describes the Witten Laplacian method for investigating decay of correlations for a class of classical lattice models in d dimensions, characterized by a continuous real local parameter and by a convex Hamiltonian Φ. The method provides a new point of view, based on PDE techniques, to approach the problem of computing and estimating thermodynamic functions in classical continuous spin models. The method can be thought as a stronger and more flexible version of the Brascamp–Lieb inequalities and is based on an exact representation of the thermodynamic functions in terms of solutions to a second order differential equation, involving a deformation of the standard Laplace–Beltrami operator. For example, it turns out that, under suitable convexity and regularity assumptions on the Hamiltonian Φ, the truncated correlation of two observables f and g can be rewritten as:

$$\langle (g-\langle g\rangle)(f-\langle f\rangle)\rangle = \int \left(\left[A_\Phi^{(1)}\right]^{-1}\nabla g\cdot\nabla h\right)e^{-\Phi(x)}dx$$

where: $x\in\mathbb{R}^\Lambda$, with $\Lambda\subset\mathbb{Z}^d$ is the lattice, and

$$A_\Phi^{(1)} = -\Delta + (\nabla\Phi)\cdot\nabla + \mathrm{Hess}\,\Phi$$

is the Helffer–Sjöstrand operator on 1-forms (equivalent to the Witten Laplacian on 1-forms). As apparent from the previous equations, exponential decay of correlations can be established in terms of properties of Φ and of the kernel $\left[A_\Phi^{(1)}\right]^{-1}$.

In this paper the author extends some previous results of Sjöstrand–Bach–Jecko and Bach–Möller on exponential decay of correlations to a larger class of convex Hamiltonians. The abstract results are illustrated by a discussion of the d dimensional Kac model, defined in terms of the potential

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{<i,j>}\log\cosh\left[\sqrt{\frac{\nu}{2}}(x_i+x_j)\right] .$$

The paper is interesting and I believe its results are suitable for publication on JSP. However I believe that the presentation should be substantially reorganized, particularly in the introductory sections. My main criticisms are the following (I will give more specific and detailed suggestions below, in the “Detailed list of comments”).

a) The first two pages of the introduction are too general and unrelated to the following. For instance the author starts by defining lattice models with a *vector* local parameter that will not be studied at all in the paper. Moreover he gives a heuristic introduction to the Boltzmann–Gibbs distribution, whose usefulness is not clear to me. I propose to cut substantially this part, directly introducing the model and notations, possibly giving a motivational introduction to models whose local parameter is *real*, as the author will assume in the following.

b) I find the heuristic description of the relation between the mean value of an observable and the solution to a Witten–Laplace equation (cf. page 3 of the paper) quite unclear. I propose to expand and make this point as clear as possible and to add references to papers giving a more systematic illustration of the relations between thermodynamic averages and Witten Laplacians PDE.

c) The comparison and the references to previous results should be made more explicit and clear. Moreover the main results of the paper should be stated clearly, in the form of a Main Theorem, in the Introduction. Presently, the main results are spread over 40 pages, inside the technical sections of the paper, and this is a very unconvenient feature of the presentation.

d) The Kac model should be motivated and references to the original papers where it was proposed should be added.

e) A more detailed description of the ideas of the proof should be given at the end of the introduction, if possible: this will help the reader to follow the proof, that occupies the following 40 pages (!).

f) Section 2 has a lot of unnecessary overlaps with the Introduction. It also includes a lengthy abstract definition of the Witten Laplacians as differential operators on a generic Riemannian manifold, that is not needed at all to the purposes of the present paper. I propose to completely eliminate this section, by cutting the abstract definition of Witten Laplacian, and to include some of its useful remarks in the introduction.

Detailed list of comments

1. p.1, l.1 to p.2, l.5 from the bottom. "In these notes...with its fixed environment". As discussed in item (a), I would substantially cut and modify this part. I would eliminate any reference to *vector* local parameters and rather give an introductory motivation to continuous spin models with local real and scalar parameter. I would eliminate the introduction to the Gibbs distribution, that I don't find useful. I would clearly introduce definitions and notation. If the Gibbs potential is defined as in the first equation of page 2, one should specify that in the following the choices $\beta = 1$ and $d\lambda(x) = dx$ will be made.
2. p.2, l.4 from the bottom: "The methods... *took* an interesting direction ... PDE techinques *are* introduced..." Please polish the english.
3. p.2, last two equations. Please make clear that the gradients and laplacians here are the usual $|\Lambda|$–dimensional gradient and laplacian. (But maybe this will not be needed if the definitions of x_i as real contiuous variables will be made clearer in the previous pages.)
4. p.3, l.7: "These later operators, provide..." → "These operators provide..."
5. p.3, l.9: "In 1996, ... equivalent to Witten Laplacians." Since the relation between the two operators is very simple, and given explicitly by Eq.(25), I would simply move Eq.(25) here.
6. p.3, l.14: "$\langle g\rangle_\Lambda$ where" → "$\langle g\rangle_\Lambda$, where"
7. p.3, 4th equation: in the denominator "e^{Φ_Λ} " → "$e^{-\Phi_\Lambda}$ ". In the line below: "for a suitable function g, ..." → "for a suitable function g with $g(0) = 0$, ..."
8. p.3, l.1 after the 6th equation: "Under suitable assumptions on the Hamiltonian, one can see..." I don't understand the idea behind the next two equations. For instance I don't understand why one can see that automatically a solution of the 5th equation of page 3 is also a solution of the 7th equation, including the presence of $\langle g\rangle_\Lambda$ in the r.h.s.: is the presence of this term automatic or must one impose proper boundary conditions on $\mathbf{v}$? Also, what are the misterious "suitable assumptions" on Φ? Smoothness? Convexity? Please be specific. Finally, by deriving the 7th equation I don't get the 5th equation, unless $\partial_i v_j = \partial_j v_i$. Is this correct? Is this symmetry a property that must be enforced on $\mathbf{v}$ or is it automatic? Please expand this part and be as clear as possible. Also, please add a reference to a book or paper where a more systematic illustration of the relations between Witten Laplacians and thermodynamic averages is presented.
9. p.3, l.5 from the bottom: "is then reduced" → "is reduced"
10. p.4, l.1: "...a stronger and more flexible version of the Brascamp-Lieb inequality." In which sense is the Witten Laplacian method stronger? Are the required convexity assumptions weaker? Are the final results stronger? Please explain.
11. p.4, l.7 from the bottom: "under under weaker assumptions" → "under weaker assumptions". Also: weaker than what?

12. p.4, l.3 from the bottom: "We attempt in this paper, to study" → "In this paper, we study"

13. p.4, last line. "We removed limitations of earlier work of Heffler and Sjostrand." And what about the results of Sjostrand-Bach-Jecko and Bach-Moller? Please comment on the relation between the results of the present paper with those of these other two papers. As discussed in item (c), please also add an explicit statement of the Main Results of this paper in the form of a Theorem. A technically precise statement of the results will also make easier to compare them with previous results.

14. p.5, 4th equation: is the + in $(x_i + x_j)$ correct? Please explain the meaning of this model: give motivations and a reference to the original paper where it was introduced.

15. p.5, l.1 after 4th equation: "ν_o" → "ν_0". Please make the same replacement all over the paper: this mistake has been repeated many times below (e.g., at p.12-13, H_o should be H_0, at p.15 $B^1_{o,\Phi}$ and C^∞_o should be $B^1_{0,\Phi}$ and C^∞_0, etc.)

16. p.5, end of Section 1. Please add some more details about the ideas of the proof, in order to guide the reader through the lengthy proof below. I would suggest to either expand the plan of the paper, or to add a subsection "Outline of proof".

17. Please change the enumeration of formulas so that the formulas in Section x are labelled by $(x.1), (x.2), ...$ Please also modify the enumeration of Lemmas and Theorems, so that the first Lemma in the paper is Lemma 1, the first Theorem is Theorem 1 and so on (at the moment the first Theorem after Lemma 1 is labelled as Theorem 2...).

18. Section 2. As discussed in item (f), I would completely elimninate this section. I would just keep Eq.(25) and following lines, moving them in the Introduction, as proposed above. If the author decides not to eliminate this section, please consider the following comments: *(i)* p.6, l.3 after Eq. (11). "convention namely" → "convention, namely". *(ii)* p.6, Eq.(13): before the equation I would add "Note that in particular". In the equation a period should be eliminated before the last Φ. *(iii)* p.6, Eq.(16). What is p? After the equation I would add "where the superscript p means..." *(iv)* p.7, Eq.(19). The partial derivative $\partial/(\partial x_k)$ acting on the last u in the r.h.s. is not displayed correctly.

19. p.8, l.1. Here the assumptions on the phase space to be used below are stated for the first time. As discussed above, please move these assumptions at the very beginning of Section 1.

20. p.8, l.2. "guaranty" → "guarantee"

21. p.8, l.1 after (26). "consists of" → "consists in" (maybe?)

22. p.8, 2nd equation. Please define α and $|\alpha|$.

23. p.8, l.4 after Eq.(27). "in a variational sense involving" → "as a variational problem involving".

24. p.8, l.8 after Eq.(27). "Compact" → "compact". Add a reference for this theorem. Similarly for the Fredholm alternative mentioned in the next line.

25. p.8, l.3 before (28). "Since, in the context" → "Since in the context"

26. p.8, l.1 after (28). "In the case where" → "in the case that"

27. p.8, l.3 after (28). "[11]lem.5." → "see Lemma 5 in [11]".

28. p.8, 1st equation of Lemma 1. Add a parenthesis in the r.h.s.: "$\theta|\nabla\Phi(x)|^2 - \Delta\Phi$" → "$\big(\theta|\nabla\Phi(x)|^2 - \Delta\Phi\big)$"

29. p.9, l.4 of the proof. "..bounded sequence H^1..." → "..bounded sequence in H^1..."

30. p.9, l.5 of the proof. "...$d\mu$).Moreover" → "...$d\mu$). Moreover"

31. p.9, l.2 before (29). "$L^2(B(0,R))$.We" → "$L^2(B(0,R))$. We" and "$\eta > 0$.The" → "$\eta > 0$. The"
32. p.9, last three equations. Please use a more compact symbol for $\inf_{\mathbb{R}^\Lambda \setminus B(0,R)}$ that is currently not displayed correctly. For instance, define $B^c \equiv \mathbb{R}^\Lambda \setminus B(0,R)$. Moreover, please modify Eq. (30)-(31) so that the numbers are displayed clearly.
33. p.10, Eq.(39): the first parenthesis should be eliminated
34. p.10, l.1 after (39): "C_o^∞" → "C_0^∞"
35. p.10, l.9 from the bottom. I don't like the sentence "...the direction towards the assumptions needed...", please polish the sentence.
36. p.10, Assumptions on Φ. These assumptions should already appear in the Introduction, in the statement of the Main Theorem, that I suggested to add. Please also add a list of examples, relevant for applications, with a Φ satisfying these assumptions.
37. p.11, l.3. "v.The" → "v. The"
38. p.11, l.4. "$1(\cdot,\cdot)$.The" → "$1(\cdot,\cdot)$. The"
39. p.11, l.4: "$a + \lambda$" → "$a + \lambda :$"
40. p.11, 5th equation: v should be eliminated from the r.h.s.
41. p.12, Theorem 2. "Lax Milgram" → "Lax, Milgram" and "there exists an u" → "there exists a u"
42. p.12, l.1 before Lemma 4. "Hilbert H" → "Hilbert space H"
43. p.13, l.2. "F.We" → "F. We" Please check that this mistake is not repeated again below.
44. p.13, l.1 after 3rd equation. "by Lax-Milgram" → "by the Lax-Milgram theorem"
45. p.13, l.1 before Theorem 5. "Summing up" → "Summarizing"
46. p.14, Theorem 6. "have solutions (non unique)" → "have (non unique) solutions"
47. p.14. Proof of Thm 6. "see" → "See"
48. p.14, Remark 7. I don't understand the role of this remark here. Maybe it should be moved to the end of previous section.
49. p.15, 4th equation. B^k_Φ has been already defined above: don't repeat the definition (at most recall where B^k_Φ was defined)
50. p.15, l.2 after 4th equation. "be the closure" → "the closure"
51. p.16, l.3 from the bottom. "theorem 1" → "theorem 2". Also, please be consistent in capitalizing (or not) the word "Theorem"
52. p.20, l.2 above Theorem 11. "use" → "uses"
53. p.20, proof of Theorem 11. "see" → "See"
54. p.21, Remark 12. Please give a name to this remark (i.e., **Remark 12 (proof of Theorem 9)**). Moreover the text of the remark should not be italic.
55. p.21, l.3 of section 6. "Mechanics and is given by" → "Mechanics, given by"
56. p.22, l.2 from the bottom. A space is missing before "denote". Moreover "balls" → "ball"
57. p.41, Acknowledgements. "in the writing" → "in writing" and "I also would like" → "I would also like".
58. p.41–46, References. Please include *only* the references actually cited in the text.