# Research Statement

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## 1 Introduction

My mathematical research interests are partial differential equation methods for the mathematical description of critical phenomena in Statistical Physics and Euclidean Field Theory.

My current research in mathematical physics involves the study of direct methods for integrals and operators of the type that appear naturally in Equilibrium Statistical Mechanics and Euclidean Field Theory. The focus is on phase transitions; the techniques come from partial differential equations. This document describes my short-term research goals, achievements, and future plans.

In my dissertation, I used the Witten Laplacians to study the decay of the correlation functions and the analyticity of the pressure for classical convex unbounded spin systems.

## 2 Background

In the context of classical equilibrium Statistical Mechanics, one is interested in a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components. Consider, for example, a piece of ferromagnetic metal (like iron, cobalt, or nickel) in thermal equilibrium. It consists of a very large number of atoms located at the sites of a crystal lattice $\Lambda$. Each atom has a magnetic moment which can be visualized as a vector in $\mathbb{R}^3$. This magnetic moment is called the *spin* of the atom and represents the orientation of the atom in the lattice. Denote by $S$ the set of all possible orientations of the spins. Each element $i$ of $\Lambda$ is called a (lattice) *site.* A particular configuration of the total system will be described by an element $x = (x_i)_{i\in\Lambda}$ of the product space $\Omega = S^{\Lambda}$. This set $\Omega$ is called the *configuration space.*

The physical system considered above is characterized by a sharp contrast: the microscopic structure is enormously complex, and any measurement of microscopic quantities is subject to statistical fluctuations. The macroscopic behavior, however, can be described by a few parameters such as magnetization and temperature, and macroscopic measurement leads to apparently deterministic results. This contrast between the microscopic and the macroscopic level is the starting point of Classical Statistical Mechanics as developed by Maxwell, Boltzman, and Gibbs. Their basic idea may be summarized as follows: The microscopic complexity may be overcome by a statistical approach, and the macroscopic determinism may then be regarded as a consequence of a suitable law of large numbers. According to this philosophy, it is not adequate to describe the state of the system by a particular element $x$ of the configuration space $\Omega$. The state should rather be described by a probability measure $\mu$ on $\Omega$ consistent with the available partial knowledge of the system. In particular, $\mu$ should take account of the a priori assumption that the system is in thermal equilibrium.

Which kind of probability measure on $\Omega$ is suitable to describe a physical system in equilibrium? The term equilibrium depends on the forces and energies that act on the system. Thus one needs to define a Hamiltonian $\Phi$ that assigns to each configuration $x$ a potential energy $\Phi(x)$. In the physical system above, the essential contribution to the potential energy comes from the interaction of the microscopic components of the system and from a possible external force. As soon as a Hamiltonian $\Phi$ has been specified, the answer to the question is generally believed to be the probability measure

$$d\mu(x) = Z^{-1}e^{-\beta\Phi(x)}d\lambda(x)$$

Here $d\lambda$ refers to a suitable a priori measure (for example the counting measure if $\Omega$ is finite), $\beta$ is a positive number that is proportional to the inverse of the absolute temperature, and $Z > 0$ is a normalization constant. The above measure $\mu$ is called *the Boltzmann-Gibbs distribution.*

The number of atoms in a ferromagnet is extremely large. Consequently, the set $\Lambda$ in our mathematical model should be very large. According to a standard rule of mathematical thinking, the intrinsic properties of large objects can be made manifest by performing suitable limiting procedures. It is therefore a common practice in Statistical Physics to pass to the infinite volume limit $|\Lambda| \to \infty$. (This limit is also referred to as the thermodynamic limit). The explicit formula for the Boltzmann-Gibbs distribution does not admit a direct extension to infinite systems. However, when dealing with infinite systems, we can still look at finite subsystems, provided the rest is held fixed. Indeed, starting with an interacting potential $\phi$ we can define for each finite subsystem $\Lambda$ a Hamiltonian $\Phi_\Lambda$ which includes the interaction of $\Lambda$ with its fixed environment. Then we can consider a limiting Gibbs measure that describes the system in an ideal infinite volume limit situation.

In the above, we argued that the physical systems like ferromagnets in equilibrium are reasonably modelled by Gibbs measures. We should then expect the Gibbs measure to exhibit a certain kind of behavior which reflects the physical phenomenon of phase transition.

In order to find out what should happen, we consider the spontaneous magnetization of a ferromagnet at low temperature. First we place the ferromagnet in an external magnetic field (which is oriented along one of the axes of the ferromagnetic crystal). Turning the field off and waiting until equilibrium, we find that the ferromagnet exhibits a macroscopic magnetic moment in the same direction as the stimulating external field. A second experiment with an external field in the opposite direction produces an equilibrium state with the opposite magnetization as before. The ferromagnet thus admits two distinct equilibrium states. We thus expect that the physical phenomenon of phase transition should be reflected by the non-uniqueness of Gibbs measures. In 1968 Roland Dobroshin, one of the founders of modern rigorous Statistical Mechanics, proposed a uniqueness condition that would imply the absence of phase transitions. The condition roughly stated that the total interaction of a given spin with all other spins should be very small. This triggered interest in the study of the exponential decay of the two-point correlation function. The study of the exponential decay of the correlation also gained much interest when Fröhlich and Spencer discovered in 1981 that the non-uniqueness of equilibrium state is not the only critical phenomenon of physical interest, but a different sort of transition is characterized by a change from an exponential decay of the correlation to a power law decay.

The methods for investigating the dynamical behavior of classical unbounded spin systems took an interesting direction when powerful and sophisticated PDE techniques were introduced. The methods are generally based on the analysis of suitable differential operators

$$\mathbf{W}_{\Phi}^{(0)} = \left( -\boldsymbol{\Delta} + \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} \right)$$

and

$$\mathbf{W}_{\Phi}^{(1)} = -\boldsymbol{\Delta} + \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} + \mathbf{Hess}\Phi$$

that are in some sense, deformations of the standard Laplace-Beltrami operator. These operators, commonly called Witten Laplacians, were first introduced by Edward Witten [13] in 1982 in the context of Morse theory for the study of topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Jöhannes Sjostrand [5] introduced two elliptic differential operators

$$A_{\Phi}^{(0)} := -\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}$$

and

$$A_{\Phi}^{(1)} := -\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla} + \mathbf{Hess}\Phi.$$

These latter operators provide direct methods for the study of integrals and operators in high dimensions for problems of the type that appear in Statistical Mechanics and Euclidean field theory. In 1996, Jöhannes Sjostrand [9] observed that these so-called Helffer-Sjostrand operators are in fact equivalent to Witten Laplacians.

Numerous techniques have been developed in the study of Laplace integrals associated to the equilibrium Gibbs state for unbounded spins systems. One of the most striking results is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one-forms that leads to sophisticated methods for estimating the correlation functions. This formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [1]. The formula may be written as follow:

$$\mathbf{cov}(f,g) = \int \left( A_{\Phi}^{(1)^{-1}} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \right) e^{-\Phi(x)} dx. \tag{1}$$

New methods that are purely based on spectral analysis have been recently developed by Helffer-Bodineau [2], Sjostrand-Bach-Jecko[14]. In these papers, the authors studied a certain class of unbounded spin models by means of the spectrum of the Witten Laplacian. In [15], the asymptotics of the two point correlation function to leading order in $\beta^{-1}$ was obtained under weaker assumptions on the Hamiltonian. In 2003 V. Bach and J S. Moller [15] proposed a refined version of the results in [15] by introducing a new twisted Witten Laplacian to relax the convexity assumption.

# 3 Results

The results obtained in my dissertation, **Witten Laplacian Methods for Critical phenomena**, include estimates leading to the exponential decay of the two point correlation functions of a random field with domain a subset $\Lambda$ of $\mathbb{Z}^d$, and values in $\mathbb{R}$.
Specifically, for a strictly convex Hamiltonian of the Kac type given by

$$\Phi(x) = \frac{x^2}{2} + \Psi(x), \qquad x = (x_i)_{i\in\Lambda}\,,$$

I proved exponential decay of correlations by using a particular technical assumption on $\Phi$. I assumed that for some $\delta > 0$ and $\kappa < +\infty$, and for every weight function $\rho : \Lambda \to \mathbb{R}^+$ satisfying

$$e^{-\kappa} \leq \frac{\rho\,(i)}{\rho(j)} \leq e^{\kappa}, \text{ if } i \text{ and } j \text{ are nearest neighbor in } \Lambda,$$

the potential $\Phi$ satisfies

$$M^{-1}\mathbf{Hess}\Phi(x)M \geq \delta,$$

where $M$ is the diagonal matrix $M$ given by

$$M = \left(\delta_{ij}\rho(j)\right)_{ij}.$$

This removes limitations of earlier work of Helffer and Sjostrand [5]. They only treated the one dimensional case ($d = 1$) under the artificial restrictions

$$\|\mathbf{Hess}\Phi(x)\|_{\mathcal{L}(l^\infty_\rho)} \leq C$$

and

$$\|\mathbf{Hess}\Phi(x) - \mathbf{I}\|_{\mathcal{L}(l^\infty_\rho)} \leq \delta < 1,$$

for all weight function $\rho$ on $\mathbb{Z}/m\mathbb{Z}$ satisfying

$$e^{-\kappa} \leq \frac{\rho\,(i+1)}{\rho(i)} \leq e^{\kappa}, \text{ for some } \kappa > 0.$$

These conditions are too restrictive for many important applications, while my conditions are considerably more flexible. In particular, the conditions in my work are suitable for treating the higher dimensional Kac model, where the potential is given by

$$\Phi(x) = \frac{x^2}{2} - \sum_{i,j\in\Lambda, i\sim j} \ln\cosh\left[\sqrt{\frac{\nu}{2}}\,(x_i + x_j)\right], \qquad x = (x_i)_{i\in\Lambda}\,,$$

for $\nu > 0$ small enough.

Another result I obtained in my dissertation is a formula suitable for a direct proof of the analyticity of the pressure for certain classical unbounded spin systems. The motivation for the study of the differentiability or even the analyticity of the pressure with respect to distinguished thermodynamic parameters such as temperature, chemical potential , or external field comes from the fact that the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition.
We obtained the formula

$$\frac{d^n}{dt^n} P_\Lambda(t) = \frac{(n-1)! < A_g^{n-1} g >_{t,\Lambda}}{|\Lambda|},$$

for the $nth$ derivative of the pressure when regarded as function of a thermodynamic parameter $t$. Here,

$$P_\Lambda(t) = \frac{1}{|\Lambda|} \log\left[\int dx e^{-\Phi^t(x)}\right],$$

where

$$\Phi^t(x) = \Phi_\Lambda(x) - tg(x),$$

$\Phi_\Lambda$, $g$ are suitable $C^\infty-$functions, and

$$A_g f := \left(A^1_{\Phi^t}\right)^{-1} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g.$$

Notice that a good estimate of $< A_g^{n-1} g >_{t,\Lambda}$ would yield a control of the coefficients in the Taylor expansion of $P_\Lambda(t)$. This gives promise for a direct method of proving analyticity of the pressure, i.e., by a method based on a direct $C^n$ bound on the $nth$ order coefficient of the Taylor expansion. The methods known up to now rely on complicated indirect arguments.

# 4 Future directions

## 4.1 In mathematical physics

The relevance of my result about the analyticity of the pressure can be seen by its potential contribution towards the solvability of the dipole gas problem in Coulomb systems. The dipole gas and other gases of particles interacting

through Coulomb forces are very important systems in Statistical Mechanics. In particular, for dipole gases, the lack of screening is well known [16], and the analyticity of the pressure in the high temperature and low activity region has been proved in an indirect way, by means of renormalization group methods (see [17] and [18]). A direct proof of the analyticity of the pressure based on estimating the coefficients of the Mayer (Taylor) series is still an open problem. The close relationship between this model and the Coulomb gas in the Kostelitz-Thouless phase ($\beta > 8\pi$), go along with the non-existence of any proof for the analyticity of the pressure in the Coulomb gas. Indirect arguments are attempted in [19],[20] and [21].
We believe that after a suitable regularization of the Coulomb potential at short distances, one can fit the problem into the framework of the models discussed in my dissertation and get an estimate of the coefficients of the Mayer series through this new formula for the derivatives of the finite volume pressure. This is one of our short term goals.

## 4.2 In Geometric Analysis

The use of the Witten Laplacians in my work in statistical physics triggered my interest in the study of the topology and geometry of Riemannian manifolds by means of differential operators. In particular, I am interested in using a family of Witten Laplacians

$$\mathbf{W}_\Phi^{(k)} = \mathbf{d}_\Phi \mathbf{d}_\Phi^* + \mathbf{d}_\Phi^* \mathbf{d}_\Phi$$

where

$$\mathbf{d}_\Phi = \mathbf{e}^{-\frac{\Phi}{2}} \mathbf{d} \mathbf{e}^{\frac{\Phi}{2}} \quad \text{and} \quad \mathbf{d}_\Phi^* = \mathbf{e}^{\frac{\Phi}{2}} \mathbf{d}^* \mathbf{e}^{-\frac{\Phi}{2}}.$$

Here, $\mathbf{d}^*$ is the exterior coderivative. These operators acting on $k-$forms on $\mathbf{M}$ are called Witten Laplacians. They were first introduced by E.Witten [13] in 1982. When $\Phi \equiv 0$, $\mathbf{W}_0^{(k)} = \mathbf{\Delta}^{(k)}$, the standard Laplace-Beltrami operator. It is well known that the spectrum $\{\lambda^{(k)}\}$ of $\mathbf{\Delta}^{(k)}$ contains both topological and geometric information. In particular, by the Hodge theorem, the dimension of the kernel of $\mathbf{\Delta}^{(k)}$ equals the $k^{th}$ Betti number, and so the Laplacians $\mathbf{\Delta}^{(k)}$ determine the Euler characteristic, which is a topological invariant of the manifold. On the other hand, if we consider the heat equation

$$\left(\partial_t + \mathbf{\Delta}^{(k)}\right) u = 0$$

on $k-$forms with solution given by the heat semigroup $e^{-t\mathbf{\Delta}^{(k)}} u_o$, $u_o$ being the initial $k-$form, the behavior of the trace of the heat semigroup

$$Tr(e^{-t\mathbf{\Delta}^{(k)}}) = \sum_i e^{-t\lambda_i^{(k)}}$$

as $t \to 0$ is controlled by an infinite sequence of geometric data, involving the volume of the manifold and the integral of the scalar curvature. In the case where the operators are given by the $\mathbf{W}_\Phi^{(k)}$'s for a suitable nonzero smooth function $\Phi$, one can hope to get more general and flexible versions of the results already obtained in the case of the Laplace-Beltrami operator where $\Phi \equiv 0$. This is already seen in analysis with the Helffer-Sjostrand formula for the covariance that provides a more general and flexible version of the Brascamp-Lieb inequality.

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