King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260
Major Exam II, Semester II, 2008-2009

Duration: 120 minutes

Name:____

ID:____

Section:____

Answer the questions in the space provided. You must show your work or explain your solution otherwise points may be deducted. If you make an unnecessary approximation in your solution to a problem, your answer will be judged on its accuracy. Points may be deducted for poor or inappropriate approximation.

1. Write clearly.
2. Show all your steps.
3. No credits will be given to wrong steps.
4. Calculators and mobile phones are NOT allowed in this exam.

Q#	Marks	Maximum Marks
1		5
2		10
3		5
4		4
5		8
6		6
7		6
8		6
Total		50

Problem1 .

Find (if possible) conditions on $b_1,\ b_2,$ and b_3 such that the system of linear equations

$$\begin{aligned} x + y + 2z &= b_1 \\ x + z &= b_2 \\ 2x + y + 3z &= b_3 \end{aligned}$$

(A) is consistent, (B) is inconsistent

(5pts) **Solution**

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right) \qquad (1pt)$$

$-R_1 + R_2,$ and $-2R_1 + R_3 \rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right)$$

$-R_2 \rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right)$$

$R_2 + R_3 \rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right) \qquad (2pts)$$

It is now evident from the 3rd row that the system is

(A) consistent if

$$b_3 - b_2 - b_1 = 0 \qquad (1pt)$$

and

(B) inconsistent if

$$b_3 - b_2 - b_1 \neq 0. \qquad (1pt)$$

Problem2 .
(A) Find

$$(AB^{-1})^{-1}$$

if

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

(7 pts) **Solution**
We have

$$(AB^{-1})^{-1} = BA^{-1}. \qquad (1pt)$$

We need to find A^{-1}.
Begin by adjoining the identity matrix to A to form the matrix

$$[A \mid I] = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right). \qquad (1pt)$$

Now using elementary row operations, rewrite this matrix in the form $[I \mid A^{-1}]$ as follows:

$-R_1 + R_2 \to$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$6R_1 + R_3 \to$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right)$$

$4R_2 + R_3 \to$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right)$$

$-R_3 \to$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

$R_2 + R_3 \rightarrow$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

$R_1 + R_2 \rightarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right). \qquad (3pts)$$

Therefore

$$A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \qquad (1pt)$$

$$(AB^{-1})^{-1} = BA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -1 \\ -9 & -9 & -3 \\ 14 & 28 & 7 \end{pmatrix} \qquad (1pt)$$

(B) Let A and B be two square matrices with the same size. Find

$$\det(A^T B^{-1} A^{-2} B^3)$$

if

$$\det A = 10, \quad \text{and} \quad \det B = 3$$

- $a)$ $\dfrac{9}{1000}$
- $b)$ 81000
- $c)$ $\dfrac{9}{10}$
- $d)$ $\dfrac{3}{10}$
- $e)$ cannot be found

Solution

$$\begin{aligned}\det(A^T B^2 A^{-1} B^{-3}) &= (\det A^T)(\det B^{-1})(\det A^{-2})(\det B^3) \\ &= \frac{\det A (\det B)^3}{(\det A)^2 \det B} \\ &= \frac{(\det B)^2}{\det A} = \frac{9}{10}\end{aligned}$$

Answer is c) (3pts)

(2pts) **Problem3 .**

(A) Determine whether the following statements are **TRUE** or **FALSE.**

1. Homogeneous linear systems have nonzero solutions __?__

2. For any square matrices A and B of the same size, $AB = 0$ implies $A = 0$ or $B = 0$. __F__ (0.

3. If V is any vector space with dimension 4, then a collection of 5 nonzero vectors in V __F__ (
could be linearly independent. __F__ $(0.5pt)$

4. For any $n \times n$ matrix A if λ is a nonzero real number, then $\det(\lambda A) = \lambda \det(A)$. __F__ $(0.5pt)$

5. If A and B are any square matrices of the same size, then $(A+B)^{-1} = A^{-1} + B^{-1}$. __F__ (0.5

(B) If A is a square matrix that satisfies

$$A^n = 0 \qquad n \geq 2,$$

show that A is singular.

(3pts) **Solution**

$$\det A^n = (\det A)^n \qquad (1pt)$$

$$(\det A)^n = 0 \qquad (0.5pt)$$

implies

$$\det A = 0 \quad \text{and} \quad A \text{ is singular.} \quad (1pt)$$

Problem 4 . Determine whether

$$\left\{-\sqrt{2},\ x,\ \cos x,\ \sin x\right\}$$

is a set of linearly independent functions.

Solution

We have

$$W(-\sqrt{2},\ x,\ \cos x,\ \sin x) = \begin{vmatrix} -\sqrt{2} & x & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \\ 0 & 0 & -\cos x & -\sin x \\ 0 & 0 & \sin x & -\cos x \end{vmatrix} \quad (1pt)$$

$$= -\sqrt{2}\begin{vmatrix} 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix} \quad (1pt)$$

$$= -\sqrt{2}\begin{vmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{vmatrix}$$

$$= -\sqrt{2}\left(\sin^2 x + \cos^2 x\right) = -\sqrt{2} \neq 0. \quad (1pt)$$

We conclude that

$$\left\{-\sqrt{2},\ x,\ \cos x,\ \sin x\right\}$$

is linearly independent. $(1pt)$

Problem 5.
(A) Let

$$\mathbf{w}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}, \quad \text{and,} \quad \mathbf{w}_3 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

be three vectors in $\mathbb{R}^3$.
Which of the following statements are **true.**

a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$ spans $\mathbb{R}^3$

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ spans $\mathbb{R}^3$

c) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$, and, $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ spans $\mathbb{R}^3$

d) $\begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$, and, $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ spans $\mathbb{R}^3$

e) None of the above

(4pts) **Solution**

$$\begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} = 107 \neq 0$$

Therefore $\left\{ \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Hence it spans $\mathbb{R}^3$.

Answer is c) $\quad (4pts)$

(B) Let A be a fixed 2×2 matrix.

If $\mathcal{M}_{2,2}$ denotes the set of all 2×2 matrices, prove or disprove that the set W of all 2×2 matrices X satisfying

$$XA = AX$$

is a subspace of $\mathcal{M}_{2,2}$.

(4pts) **Solution**

Let X_1 and X_2 be two elements of W and c a scalar. We have

$$\begin{aligned}(X_1 + X_2)A &= X_1A + X_2A \\ &= AX_1 + AX_2 \\ &= A(X_1 + X_2) \qquad (2pts)\end{aligned}$$

and

$$\begin{aligned}(cX_1)A &= c(X_1A) \\ &= cAX_1 \\ &= A(cX_1) \qquad (1pt)\end{aligned}$$

Therefore W is a subspace of $\mathcal{M}_{2,2}$. $\qquad (1pt)$

Problem 6.

Determine a basis and the dimension of the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

provided that its augmented coefficient matrix reduces to

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(6pts) **Solution**

After performing the back substitution we obtain

$$\begin{aligned} x_1 + x_2 + x_5 &= 0 \\ x_3 + x_5 &= 0 \ . \qquad (1pt) \\ x_4 &= 0 \end{aligned}$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 - x_5 \\ x_2 \\ x_3 \\ 0 \\ x_5 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \\ 0 \\ -x_3 \end{pmatrix}}_{(1pt)} = \underbrace{\begin{pmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \\ x_3 \\ 0 \\ -x_3 \end{pmatrix}}_{(2pts)}$$

$$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad (1pt)$$

which shows that the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

span the solution space ($1pt$). Since they are linearly independent, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the solution space and the dimension is 2 ($1pt$).

Problem 7. Find the general solution of the second order differential equation

$$2y'' - 5y' - 3y = 0$$

Solution

The characteristic equation is

$$2r^2 - 5r - 3 = 0. \qquad (2pts)$$

Its roots are

$$r_1 = \frac{-1}{2}, \text{ and } r_2 = 3. \qquad (1pt)$$

The general solution is

$$y = c_1 e^{-x/2} + c_2 e^{3x}. \qquad (2pts)$$

Problem 8. Consider the differential equation

$$y'' - 10y' + 25y = 0$$

(a) Find the general solution
(b) Use part (a) with the initial conditions

$$y(0) = -1, \quad \text{and} \quad y'(0) = 7$$

to find a particular solution by **Cramer's rule**.

Solution
(a) The characteristic equation

$$r^2 - 10r + 25 = (r-5)^2 = 0 \qquad (1pts)$$

has 5 as a repeated root. The general solution is

$$y = c_1 e^{5x} + c_2 x e^{5x}. \qquad (1pt)$$

(b)

$$y'(x) = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}. \ (1pt)$$

$$y(0) = -1, \quad \text{and} \quad y'(0) = 7$$

gives

$$\begin{aligned} c_1 + 0c_2 &= -1 \\ 5c_1 + c_2 &= 7 \end{aligned} \qquad (1pt)$$

By Cramer's rule

$$c_1 = \frac{\begin{vmatrix} -1 & 0 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix}} = \frac{-1}{1} = -1, \qquad c_2 = \frac{\begin{vmatrix} 1 & -1 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix}} = \frac{12}{1} = 12. \qquad (2pts)$$

The corresponding particular solution is

$$y = e^{5x} + 12xe^{5x} \qquad (1pt)$$