King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Semester I, 2005-2006 (Term 051)
MATH 260-2-4 Major Exam I
Instructor: Dr. A. Shawky Ibrahim Duration: 90 Minutes

Name: KEY Serial #:

ID #: Sec. #:

- All kinds of calculators are NOT allowed in this exam.
- Write clear steps.

Question #	Marks	Maximum Marks
1		5
2		4
3		6
4		4
5		10
6		10
7		7
8		4
Bonus Problem		10
		Out of 50

MATH 260 Exam I (051)

1. (5-points) Find a differential equation that can serve as a mathematical model for the two-parameter family of curves

$$y = \alpha x^3 - \beta x^5, \quad (1)$$

where α and β are independent arbitrary constants

$$(1) \Rightarrow \frac{dy}{dx} = 3\alpha x^2 + 5\beta x^4 \quad (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6\alpha x + 20\beta x^3 \quad (3)$$

$$x(2) - 3(1) \Rightarrow x\frac{dy}{dx} - 3y = 2\beta x^5 \quad (4)$$

$$x(3) - 2(2) \Rightarrow x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 10\beta x^4 \quad (5)$$

$$(4) \,\&\, (5) \Rightarrow x\left(x\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\right) = 5\left(x\frac{dy}{dx} - 3y\right)$$

$\Rightarrow$ The required DE is

$$\Rightarrow x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 15y = 0.$$

2. (4-points) Show how the following differential equation can be solved by the method of separation of variables:

$$\frac{dy}{dx} = \frac{x^2 y e^{-2x} - 2xy + 3x^2 e^{-2x} - 6x}{y - 2}$$

[**Do not solve the differential equation**]

$$\Rightarrow \frac{dy}{dx} = \frac{y(x^2e^{-2x} - 2x) + 3(x^2e^{-2x} - 2x)}{y-2}$$

$$= \frac{(x^2e^{-2x} - 2x)(y+3)}{y-2}.$$ Now separate

variables $\Rightarrow \left(\frac{y-2}{y+3}\right) dy = (x^2e^{-2x} - 2x)\, dx.$

MATH 260 Exam I (051)

3. (6-points) The rate at which the population P of a country grows at a certain time is proportional to the total population of the country at that time. If the population increased by 30% in ten years, find the exact value of the constant of proportionality.

We have $\frac{dP}{dt} = kP \Rightarrow P = ce^{kt}$

Let $P(0) = P_0 \Rightarrow c = P_0 \Rightarrow$

$$P = P_0 e^{kt}$$

We have $P(10) = P_0 + \frac{30}{100}P_0 = \frac{13P_0}{10} \Rightarrow$

$$\frac{13P_0}{10} = P_0 e^{10k} \Rightarrow e^{10k} = 1.3 \Rightarrow$$

The exact value of the constant of proportionality

is $k = \frac{1}{10}\ln(1.3)$.

4. (4-points) Use a suitable substitution to reduce the order of the second-order differential equation

$$2y\,y'' + 5(y')^2 = 7y\,y'. \quad (1)$$

[Find the new equation but do not solve it]

Let $u = y' \Rightarrow \frac{du}{dy} = \frac{d}{dx}(y') \cdot \frac{dx}{dy}$

$\Rightarrow \frac{du}{dy} = (y'')\left(\frac{1}{y'}\right) = \frac{y''}{u}$ (2)

$\Rightarrow$ The required DE is $2yu\frac{du}{dy} + 5u^2 = 7uu'$.

MATH 260 Exam I (051)

5. (10-points) Solve Bernoulli's differential equation

$$3(1-x^2)\frac{dy}{dx} = 2xy\,(y^3-1).$$

$$\Rightarrow \quad 3(1+x^2)\frac{dy}{dx} + 2xy = 2xy^4$$

$$\Rightarrow \quad 3(1+x^2)\, y^{-4}\frac{dy}{dx} + 2x\,y^{-3} = 2x \qquad (1)$$

Put $u = y^{-3} \Rightarrow \dfrac{du}{dx} = -3y^{-4}\dfrac{dy}{dx}$ $\qquad (2)$

(1) & (2) $\Rightarrow \quad -(1+x^2)\dfrac{du}{dx} + 2xu = 2x \Rightarrow$

$$\frac{du}{dx} - \frac{2x}{1+x^2}\,u = -\frac{2x}{1+x^2} \qquad (3)$$

which is a first-order linear DE in standard form

$$\text{I.F. of (3)} = e^{-\int \frac{2x}{1+x^2}dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$$

$$\Rightarrow \quad \frac{1}{1+x^2}\,u = \int \left(\frac{1}{1+x^2}\right)\left(\frac{-2x}{1+x^2}\right)dx + C$$

$$\Rightarrow \quad \frac{1}{1+x^2}\,u = -\int (1+x^2)^{-2}(2x\,dx) + C$$

$$= (-)(-1)(1+x^2)^{-1} + C \Rightarrow$$

$$u = \boxed{y^{-3} = 1 + c(1+x^2)}$$

is the required G.S.

Math 260 Exam I (051)

6. (10-points) Show that the given differential equation is exact, then solve the initial-value problem

$$(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0, \quad y(0) = 1$$

$M = \sin y - y\sin x \Rightarrow \dfrac{\partial M}{\partial y} = \cos y - \sin x$

$N = \cos x + x\cos y - y \Rightarrow \dfrac{\partial N}{\partial x} = -\sin x + \cos y$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ the given eqn is Exact DE.

$\Rightarrow$ The G.S. is $f(x,y) = C$ where

$\dfrac{\partial f}{\partial x} = M = \sin y - y\sin x$ (i)

& $\dfrac{\partial f}{\partial y} = N = \cos x + x\cos y - y$ (ii)

(i) $\Rightarrow f(x,y) = x\sin y + y\cos x + g(y)$ (iii)

(iii) $\Rightarrow \dfrac{\partial f}{\partial y} = x\cos y + \cos x + g'(y)$ (iv)

compare (ii) & (iv) $\Rightarrow g'(y) = -y \Rightarrow g(y) = -\frac{1}{2}y^2$

$\Rightarrow f(x,y) = x\sin y + y\cos x - \frac{1}{2}y^2$

$\Rightarrow$ The G.S. is

$x\sin y + y\cos x - \frac{1}{2}y^2 = C$.

Math 260 Exam I (051)

7. (7-points) Let α be a constant. Write the augmented matrix of the linear system

$$\begin{aligned} x + 2y - 2z &= 4 \\ 3x - y + 4z &= 2 \\ 4x + y + (\alpha^2 - 14)z &= \alpha + 2 \end{aligned}$$

Then use the Gaussian elimination method to find the values of α for which the system is

(a) Inconsistent

(b) Consistent with a unique solution

(c) Consistent with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ \boxed{3} & -1 & 4 & 2 \\ \boxed{4} & 1 & \alpha^2-14 & \alpha+2 \end{array}\right] \xrightarrow[-4R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & -7 & 10 & -10 \\ 0 & \boxed{-7} & \alpha^2-6 & \alpha-14 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 4 \\ 0 & -7 & 10 & -10 \\ 0 & 0 & \alpha^2-16 & \alpha-4 \end{array}\right]$$

At this stage we can see that :

(i) the system is inconsistent if $\alpha = -4$, in such a case the last row is $0 \quad 0 \quad 0 \mid -8 \Rightarrow$ $0 = -8$ which is impossible.

(ii) the system has a unique solution if $\alpha \neq 4, -4$

(iii) the system has infinitely many solutions if $\alpha = 4$, in such a case the elements in the last row becomes all zeros and hence y and x can be written in terms of z.

Math 260 Exam I (051)

8. (4-points) Use Gauss-Jordan Elimination to find, if any, the set of all nontrivial solutions of the system

$$\begin{aligned} 2x - y + 2z &= 0 \\ 3x + 2y - z &= 0 \\ x + 3y - 3z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 3 & 2 & -1 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ \boxed{3} & 2 & -1 & 0 \\ \boxed{2} & -1 & 2 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & \boxed{-7} & 8 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = \alpha$, where α is any real number $\Rightarrow$

R_2: $-7y + 8\alpha = 0 \Rightarrow y = \frac{8}{7}\alpha \Rightarrow$

R_1: $x + 3\left(\frac{8}{7}\alpha\right) - 3(\alpha) = 0 \Rightarrow$

$x = -\frac{3}{7}\alpha$

$\Rightarrow$ The system has infinitely many solutions, and the system consist of all ordered triples of the form $\left(-\frac{3}{7}\alpha, \frac{8}{7}\alpha, \alpha\right)$ for any real number α.

Math 260 Exam I (051)

Bonus Problem (10 points) Solve the differential equation

$$x\frac{dy}{dx} + y(\ln y - \ln x - 1) = 0.$$

$\Longrightarrow x\dfrac{dy}{dx} + y\left(\ln\left(\dfrac{y}{x}\right) - 1\right) = 0.$

$\Longrightarrow y\left(\ln\left(\dfrac{y}{x}\right) - 1\right)dx + x\,dy = 0$ ①

$\Longrightarrow$ Both $M = y\left(\ln\dfrac{y}{x} - 1\right)$ and $N = x$

are homogeneous of degree 1.

Let $\dfrac{y}{x} = v \Longrightarrow y = xv \Longrightarrow$

$dy = x\,dv + v\,dx \Longrightarrow$ DE ① becomes

$xv(\ln v - 1)\,dx + x(x\,dv + v\,dx) = 0$

$\Longrightarrow [xv\ln v - \cancel{xv} + \cancel{xv}]\,dx + x^2\,dv = 0$

$\Longrightarrow xv\ln v\,dx + x^2\,dv = 0 \Longrightarrow$

$\dfrac{1}{x}\,dx + \dfrac{1}{v\ln v}\,dv = 0 \Longrightarrow$

$\displaystyle\int\frac{1}{x} + \int\frac{1}{v\ln v}\,dv = C \Longrightarrow$

$\ln|x| + \ln|\ln v| = C \Longrightarrow$ The G.S is

$$\boxed{\ln|x| + \ln\left|\ln\frac{y}{x}\right| = C}$$