KFUPM	SEM II (Term 062)	Name:	Serial #:
MATH 102	Quiz # 5	ID: #: KEY	Section #:

1. (3-points) Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{4}, +\frac{5}{8}, \frac{7}{12}, +\frac{9}{16}, \frac{11}{20}, \dots\right\}.$$

Then determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{2n+1}{4n}, \quad n \geq 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{4n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{4n}\right) = \frac{1}{2}$$

$\Rightarrow$ The sequence converges to $\frac{1}{2}$.

2. (4-points) Find the values of x for which the series $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{3^{n+1}}$ converges. Then, find the sum for those values of x.

The series is

$$\frac{(2x-1)^2}{3^3} + \frac{(2x-1)^3}{3^4} + \frac{(2x-1)^4}{3^5} + \cdots$$

which is a geometric series with first term $a = \frac{(2x-1)^2}{3^3}$ and common ratio r is

$$r = \frac{2x-1}{3}.$$

Therefore, the series is convergent if $|r| < 1 \Rightarrow \left|\frac{2x-1}{3}\right| < 1$

$\Rightarrow -3 < 2x-1 < 3 \Rightarrow$

$-2 < 2x < 4 \Rightarrow -1 < x < 2$

and for those values of x the series has the sum $= \frac{a}{1-r}$

$$= \frac{\frac{(2x-1)^2}{3^3}}{1-\left(\frac{2x-1}{3}\right)} = \frac{(2x-1)^2}{9(4-2x)},$$

$-1 < x < 2$.

P.T.O $\longrightarrow$

3. (4-points) Show that the integral test can be used to test the series $\sum_{n=1}^{\infty} \frac{2}{n^3}$ for convergence or divergence. If it converges, then find an upper bound for the size of the error if its sum S is approximated by S_{100} (write your answer in a decimal form).

Let $f(n) = \frac{2}{n^3}$, $n \geq 1 \Rightarrow$

$f(x) = \frac{2}{x^3}$, $x \geq 1$ which is a continuous and positive function for all $x \geq 1$.

$f'(x) = \frac{-6}{x^4} < 0$ for all $x \geq 1$

$\Rightarrow f$ is decreasing for all $x \geq 1$

$\Rightarrow$ The integral test can be used.

$$\int_1^{\infty} \frac{2}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t^2} + 1\right] = 1$$

$\Rightarrow$ The series is convergent.

$$R_{100} \leq \int_{100}^{\infty} \frac{2}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x^2}\right]_{100}^t = \frac{1}{(100)^2}$$

$$= \frac{1}{10{,}000} = 0.0001$$

which is an upper bound for the size of the error.

4. (4-points) Find a closed form for the general form S_n of the sequence of partial sums of the series $\sum \frac{1}{n^2 + 9n + 20}$. Then find, if possible, the sum of the series.

$$\frac{1}{n^2+9n+20} = \frac{1}{(n+4)(n+5)}$$

$$= \frac{1}{n+4} - \frac{1}{n+5}$$

$$\Rightarrow S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$= \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \left(\frac{1}{n+4} - \frac{1}{n+5}\right)$$

$$= \frac{1}{5} - \frac{1}{n+5}$$

(A telescoping sum)

Therefore, $\frac{1}{5} - \frac{1}{n+5}$ is the closed form for S_n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{5} - \frac{1}{n+5}\right)$$

$$= \frac{1}{5}$$

$\Rightarrow$ The series is convergent and has the sum $\frac{1}{5}$.