MATH 102-04-08 Exam II (052)

KEY

Part 1 Solve the following 6 Problems

1. (4- Points) The region enclosed between the curve $y = \sin x$ and the $x-$axis, for $0 \le x \le \pi$, is revolved about the $y-$axis. Find the volume of the solid generated.

$$dV \overset{\text{shell}}{=} 2\pi x \sin x \, dx$$

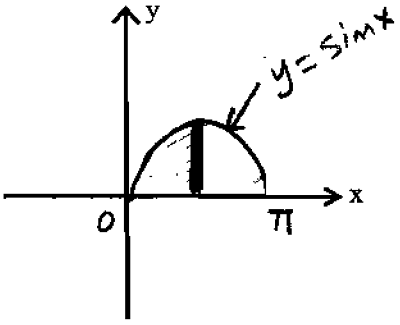
$$\Rightarrow \quad V = \int_0^{\pi} 2\pi x \sin x \, dx$$

$$\boxed{\begin{array}{ccc} x & + & \sin x \\ 1 & - & -\cos x \\ 0 & & -\sin x \end{array}}$$

$$= 2\pi \left[-x \cos x + \sin x \right]_0^{\pi}$$

$$= 2\pi(-\pi \cos\pi + 0) - (0)$$

$$= 2\pi^2.$$



2. (5- Points) Evaluate $\int_{-2}^{2} \frac{3-2x}{x^2+4x+20} dx$ $= I$

$$x^2 + 4x + 20 = (x+2)^2 + 16$$

Put $x + 2 = 4\tan\theta \Rightarrow$

$$dx = 4\sec^2\theta \, d\theta$$

$$x = -2 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$$

$$x = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Also $\quad -2x - 4 = -8\tan\theta$

$$\Rightarrow \quad 3 - 2x = 7 - 8\tan\theta$$

$$\Rightarrow I = \int_0^{\pi/4} \frac{7 - 8\tan\theta}{16\sec^2\theta} 4\sec^2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/4} (7 - 8\tan\theta) \, d\theta$$

$$I = \frac{1}{4}\left[7\theta - 8\ln|\sec\theta|\right]_0^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{7\pi}{4} - 8\ln\sqrt{2}\right) - 0\right]$$

$$= \frac{7\pi}{16} - \ln 2.$$

3. (5- Points) Find the area of the surface generated by revolving the parametric curve $x=\cos^3 t,\ y=\sin^3 t,\ 0\le t\le \pi/2$ about the x-axis.

$$\frac{dx}{d\theta} = -3\cos^2 t \sin t$$

$$\frac{dy}{d\theta} = 3\sin^2 t \cos t \quad \Longrightarrow$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t$$

$$= 9\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)$$

$$= 9\cos^2 t \sin^2 t$$

$$\Rightarrow dS = 2\pi\, y\, dL$$

$$= 2\pi \sin^3 t \sqrt{9\cos^2 t \sin^2 t}\, dt$$

$$= 6\pi \sin^4 t \cos t\, dt$$

$$S = \int_0^{\pi/2} 6\pi \sin^4 t \cos t\, dt$$

$$= \frac{6\pi}{5}\left[\sin^5 t\right]_0^{\pi/2}$$

$$= \frac{6\pi}{5}.$$

4. (5- Points) Evaluate $\int e^{-3x}\sin 2x\, dx$. $= I$

$$u = e^{-3x}, \quad dv = \sin 2x\, dx$$

$$du = -3e^{-3x}\, dx, \quad v = -\tfrac{1}{2}\cos 2x$$

$$\Longrightarrow$$

$$I = -\tfrac{1}{2} e^{-3x}\cos 2x - \tfrac{3}{2}\int e^{-3x}\cos 2x\, dx \quad ①$$

Let $J = \int e^{-3x}\cos 2x\, dx$

$$u = e^{-3x}, \quad dv = \cos 2x\, dx$$

$$du = -3e^{-3x}, \quad v = \tfrac{1}{2}\sin 2x$$

$$\Rightarrow J = \tfrac{1}{2} e^{-3x}\sin 2x + \tfrac{3}{2}\int e^{-3x}\sin 2x\, dx$$

$$= \tfrac{1}{2} e^{-3x}\sin 2x + \tfrac{3}{2} I \quad ②$$

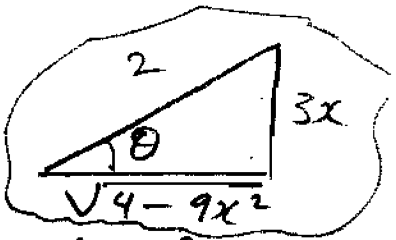
① & ② $\Longrightarrow$

$$I = -\tfrac{1}{2} e^{-3x}\cos 2x - \tfrac{3}{4} e^{-3x}\sin 2x - \tfrac{9}{4} I \quad \Longrightarrow$$

$$\frac{13 I}{4} = -\tfrac{1}{2} e^{-3x}\cos 2x - \tfrac{3}{4} e^{-3x}\sin 2x$$

$$\Longrightarrow$$

$$I = -\frac{2}{13} e^{-3x}\cos 2x - \frac{3}{13} e^{-3x}\sin 2x + C.$$



5. (5- Points) Evaluate $\int \frac{5x^2}{(4-9x^2)^{3/2}}dx.$ $= I$

Put $x = \frac{2}{3}\sin\theta \Rightarrow$

$dx = \frac{2}{3}\cos\theta \, d\theta \Rightarrow$

$$I = \int \frac{\frac{20}{9}\sin^2\theta}{(4\cos^2\theta)^{3/2}} \frac{2}{3}\cos\theta d\theta$$

$$= \frac{40}{27}\int \frac{\sin^2\theta \cos\theta}{8\cos^3\theta} d\theta \Rightarrow$$

$$I = \frac{5}{27}\int \tan^2\theta \, d\theta$$

$$= \frac{5}{27}\int(\sec^2\theta - 1)d\theta$$

$$= \frac{5}{27}[\tan\theta - \theta] + C$$

$$= \frac{5}{27}\left[\frac{3x}{\sqrt{4-9x^2}} - \sin^{-1}\frac{3x}{2}\right] + C$$

$$= \frac{5x}{9\sqrt{4-9x^2}} - \frac{5}{27}\sin^{-1}\frac{3x}{2} + C$$

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6. (5-Points) Evaluate $\int \frac{8x^2+5x+82}{(x+1)(x^2+16)}dx.$ $= I$

$$\frac{8x^2+5x+82}{(x+1)(x^2+16)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+16}$$

$\Rightarrow$

$$8x^2+5x+82 = A(x^2+16) + (Bx+C)(x+1)$$

$\boxed{x=-1}$ $85 = 17A \Rightarrow \boxed{A=5}$

$\boxed{\text{coef } x^2}$ $8 = A+B \Rightarrow \boxed{B=3}$

$\boxed{x=0}$ $82 = 16A + C = 80 + C$

$\Rightarrow \boxed{C=2} \Rightarrow$

$$I = \int\left(\frac{5}{x+1} + \frac{3x+2}{x^2+16}\right)dx$$

$$\Rightarrow I = 5\ln|x+1| + 3\int\frac{x}{x^2+16}dx$$

$$+ 2\int\frac{1}{x^2+16}dx$$

$$= 5\ln|x+1| + \frac{3}{2}\ln(x^2+16)$$

$$+ \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right) + C$$

Part II Solve 3 Problems only from the following 4 problems.

$$\begin{array}{r} 4u^2-4 \\ u^2+1\,\big)\,\overline{4u^4} \\ 4u^4+4u^2 \\ \hline -4u^2 \\ -4u^2-4 \\ \hline 4 \end{array}$$

A. (5-Points) Evaluate $\int_0^1 \frac{\sqrt[4]{x}}{1+\sqrt{x}}\,dx$ $= I$

Put $x = u^4 \Rightarrow \sqrt[4]{x} = u$ and

$\sqrt{x} = u^2$, $dx = 4u^3\,du$

$x = 0 \Rightarrow u = 0$ & $x = 1 \Rightarrow u = 1$

$$\Rightarrow I = \int_0^1 \frac{4u^4}{u^2+1}\,du$$

$$\Rightarrow I = \int_0^1 \left(4u^2 - 4 + \frac{4}{u^2+1}\right) du$$

$$= \frac{4}{3}u^3 - 4u + 4\tan^{-1}u \Big]_0^1$$

$$= \frac{4}{3} - 4 + 4\left(\frac{\pi}{4}\right)$$

$$= -\frac{8}{3} + \pi \,.$$

B. (5- Points) Use the substitution $u = \tan\frac{x}{2}$ to evaluate $\int \frac{\cos x}{\sin x(1+\cos x)}\,dx.$ $= I$

$\sin x = \frac{2u}{1+u^2}$) $\cos x = \frac{1-u^2}{1+u^2}$

$dx = \frac{2}{1+u^2}\,du \Rightarrow$

$$I = \int \frac{\left(\frac{1-u^2}{1+u^2}\right)\left(\frac{2\,du}{1+u^2}\right)}{\frac{2u}{1+u^2}\left(1+\frac{1-u^2}{1+u^2}\right)}$$

$$= \int \frac{2(1-u^2)\;du}{2u\,(1+u^2+1-u^2)}$$

$$= \frac{1}{2}\int\left(\frac{1-u^2}{u}\right)du$$

$$= \frac{1}{2}\int\left[\frac{1}{u} - u\right]du$$

$\Rightarrow$

$$I = \frac{1}{2}\ln|u| - \frac{1}{4}u^2 + C$$

$$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\left(\tan\frac{x}{2}\right)^2 + C$$

C. (i) (2.5- Points) Given $\sinh x = \frac{3}{4}$, Find the value of $\sinh 2x$ [Write your answer in simplest form]

$$\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow \cosh x = \frac{5}{4} \Rightarrow$$

$$\sinh 2x = 2 \sinh x \cosh x = 2\left(\frac{3}{4}\right)\left(\frac{5}{4}\right) = \frac{15}{8}.$$

(ii) (2.5- Points) Evaluate $\int_0^{\ln 2} \tanh x \, dx$. [Write your answer in simplest form]

$$= \int_0^{\ln 2} \frac{\sinh x}{\cosh x} dx = \left[\ln \cosh x\right]_0^{\ln 2} = \ln\left[\cosh \ln 2 - \ln 1\right]$$

$$= \ln\left[\frac{e^{\ln 2} + e^{-\ln 2}}{2}\right] = \ln\left[\frac{2+\frac{1}{2}}{2}\right] = \ln\frac{5}{4}.$$

D. (5-Points) Evaluate $\int \frac{6x^2-5x+5}{(x+1)(x-1)^2} dx$ $= I$

$$\frac{6x^2-5x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$6x^2-5x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$\boxed{x=-1}$ $16 = 4A \Rightarrow \boxed{A=4}$

$\boxed{x=1}$ $6 = 2C \Rightarrow \boxed{C=3}$

$\boxed{\text{Cof } x^2}$ $6 = A + B \Rightarrow \boxed{B=2}$

$$\Rightarrow I = \int \frac{4dx}{x+1} + \int \frac{2dx}{x-1} + \int \frac{3dx}{(x-1)^2}$$

$$\Rightarrow I = 4\ln|x+1| + 2\ln|x-1 - \frac{3}{x-1} + C$$