MATH 102-4-8 EXAM I (052) KEY



1. (4-points) Evaluate $\int \frac{2x+1}{\sqrt{x-2}}\,dx. = I$

Put $u = \sqrt{x-2} \Rightarrow du = \frac{1}{2\sqrt{x-2}}\,dx$ and $x = u^2+2$

$$\Rightarrow I = \int \left[2(u^2+2)+1\right](2\,du)$$

$$= 2\int (2u^2+5)\,du$$

$$= \frac{4}{3}u^3 + 10u + C$$

$$= \frac{4}{3}(x-2)^{3/2} + 10(x-2)^{1/2} + C$$

Note: You may use the substitution $u = x-2$ ... etc

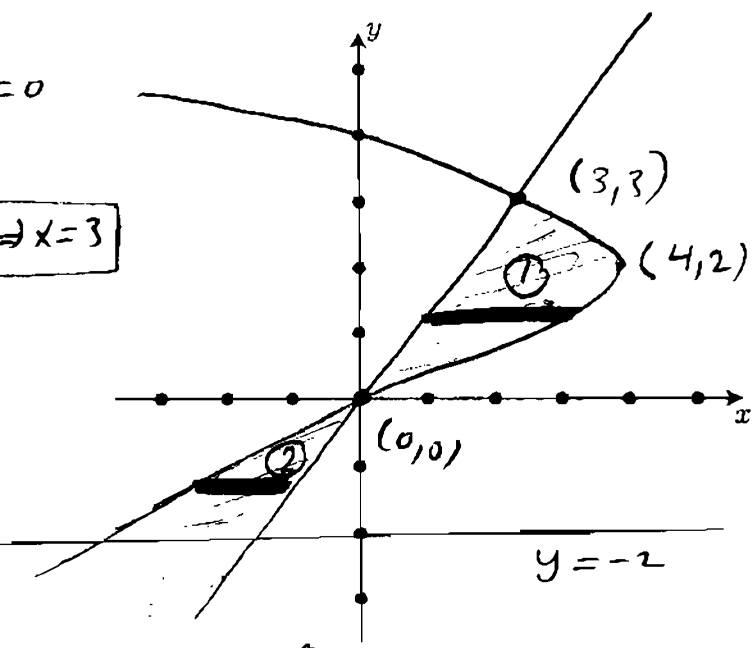
2. (4-points) Set up an integral or a sum of integrals that represents the total area between the graphs of $x = 4y - y^2$ and $y = x$ from $y = -2$ to $y = 3$. **[Do Not Evaluate the Integral(s)].**

pts of intersection

$4y - y^2 = y \Rightarrow y^2 - 3y = 0$

$y(y-3) = 0 \Rightarrow$

$\boxed{y=0 \Rightarrow x=0}$, $\boxed{y=3 \Rightarrow x=3}$




$A = A_1 + A_2$

$dA_1 = (4y - y^2 - y)\,dy$

$dA_2 = (y - (4y - y^2))\,dy$

$$\Rightarrow A = \int_0^3 (3y - y^2)\,dy + \int_{-2}^0 (y^2 - 3y)\,dy$$

3. (4-points) Evaluate $\int \frac{2x^3+3x^2-5}{x+1}\,dx$.

after Long Division $\int \left(2x^2+x-1-\frac{4}{x+1}\right)dx$

$= \frac{2}{3}x^3+\frac{1}{2}x^2-x-4\ln|x+1| + C$

Division

$$\begin{array}{r|l} & 2x^2+x-1 \\ x+1 & 2x^3+3x^2\quad -5 \\ & 2x^3+2x^2 \\ \hline & \quad x^2\quad -5 \\ & \quad x^2+x \\ \hline & \quad -x-5 \\ & \quad -x-1 \\ \hline & \quad -4 \end{array}$$

or use synthetic Division

$$\begin{array}{r|rrrr} -1 & 2 & 3 & 0 & -5 \\ & & -2 & -1 & 1 \\ \hline & 2 & 1 & -1 & -4 \end{array}$$

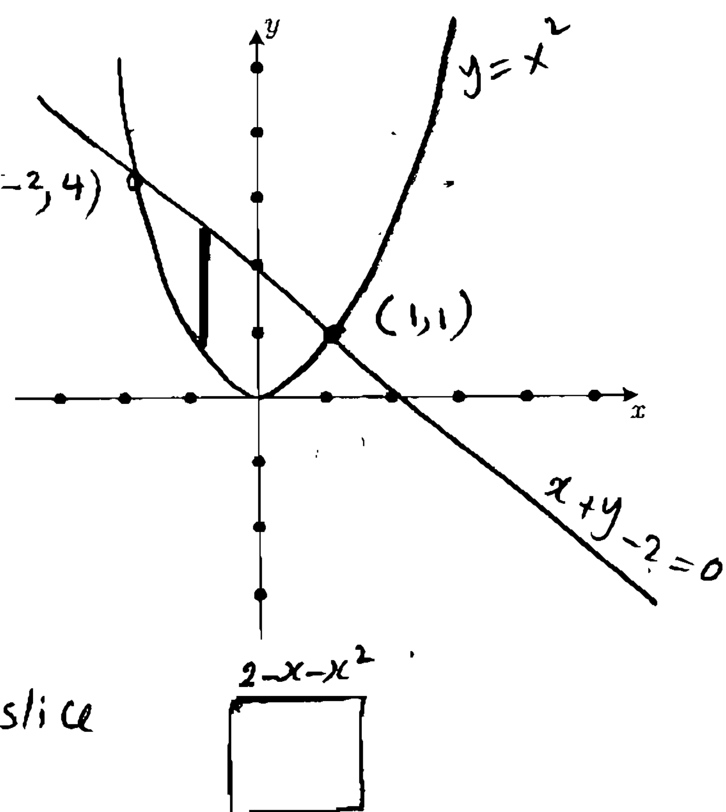
4. (4-points) The base of a solid is the region between the graphs of $y = x^2$ and $x + y - 2 = 0$. The cross-sections perpendicular to the $x$-axis are squares with bases running from one curve to the other. Set up an integral that represents the volume of the solid.

Pts of intersection

$x + x^2 - 2 = 0 \Rightarrow$

$(x+2)(x-1) = 0$

$x=-2,\ x=1$




The length of a side of the square $= y_{upper} - y_{lower}$

$= [(2-x) - x^2]$

$\Rightarrow dA = (2-x-x^2)^2\,dx$

= Area of a typical slice

$\Rightarrow A = \int_{-2}^{1} (2-x-x^2)^2\,dx$

5. (4-points) Evaluate $\int \frac{e^{-x}}{1+e^{x}}\,dx.$ $= I$

Put $u = e^{-x} \Rightarrow du = -e^{-x}dx$ and $e^{x} = \frac{1}{u} \Rightarrow$

$$I = \int \frac{-du}{1+\frac{1}{u}} = \int \frac{-u}{u+1}\,du$$

$$= -\int \left(\frac{u+1-1}{u+1}\right) du \quad \text{(or use long division)}$$

$$= -\int \left(1 - \frac{1}{u+1}\right) du$$

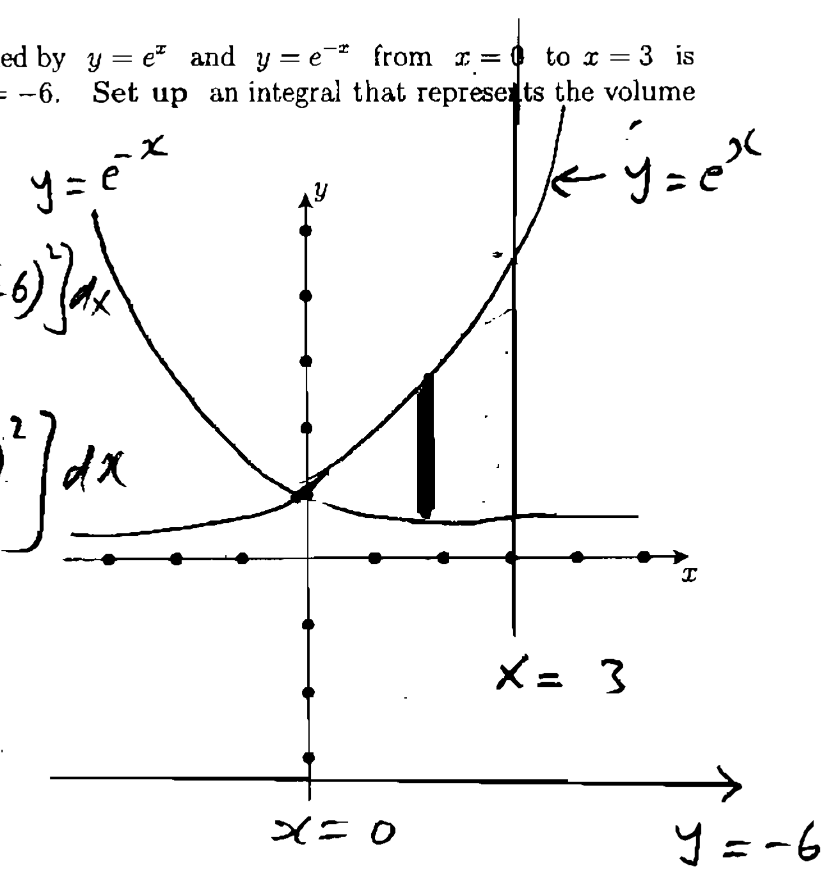
$$= -u + \ln|u+1| + C$$

$$= -e^{-x} + \ln|e^{-x}+1| + C$$

6. (4-points) The region bounded by $y = e^{x}$ and $y = e^{-x}$ from $x = 0$ to $x = 3$ is revolved about the line $y = -6$. **Set up** an integral that represents the volume of the solid generated.

$$dV \underset{\text{washer}}{=} \pi\left[(e^{x}+6)^2 - (e^{-x}+6)^2\right]dx$$

$$V = \pi \int_0^3 \left[(e^{x}+6)^2 - (e^{-x}+6)^2\right] dx$$



7. (4-points) Evaluate $\int \frac{\cos x}{\sqrt{9-4\sin^2 x}}\,dx. = I$

Put $u = 2\sin x \Rightarrow du = 2\cos x\,dx$

$$I = \int \frac{\frac{1}{2}\,du}{\sqrt{9-u^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2}{3}\sin x\right) + C$$

8. (4-points) Find the exact average value of the function $f(x) = \frac{\ln x}{x}$ over the interval $[e, e^2]$.

$$\text{average of } f = \frac{1}{e^2-e}\int_e^{e^2} \frac{\ln x}{x}\,dx$$

Put $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$x = e \Rightarrow u = \ln e = 1$, $x = e^2 \Rightarrow u = \ln e^2 = 2$

$$\Rightarrow \text{ave } f = \frac{1}{e^2-e}\int_1^2 u\,du = \frac{1}{e^2-e}\left[\frac{1}{2}u^2\right]_1^2$$

$$= \frac{2-\frac{1}{2}}{e^2-e} = \frac{3}{2(e^2-e)}.$$

9. (4-points) Find the slope of the tangent line to the graph of $F(x) = (\ln 2)^2 \int_{1/x}^{3/x} e^{-1/t}\, dt$ at $x = \ln 8$. [**Write your answer in simplest form**]

$$F'(x) = (\ln 2)^2 \left[ e^{-\frac{x}{3}}\left(-\frac{3}{x^2}\right) - e^{-x}\left(-\frac{1}{x^2}\right) \right].$$

$$= -\frac{(\ln 2)^2}{x^2}\left[ -3e^{-\frac{x}{3}} - e^{-x} \right]$$

The slope $= F'(\ln 8) = F'(3\ln 2)$

$$= -\frac{(\ln 2)^2}{9(\ln 2)^2}\left[ 3e^{-\ln 2} - e^{-\ln 8} \right]$$

$$= -\frac{1}{9}\left[ \frac{3}{2} - \frac{1}{8} \right] = -\frac{1}{9}\left[ \frac{11}{8} \right]$$

$$= \frac{-11}{72}.$$

10. (4-points) If $\int_3^7 f(x)\, dx = 11$, find the value of $\int_3^{15} f\left(\frac{1}{3}x + 2\right) dx$.

Put $u = \frac{1}{3}x + 2 \Rightarrow du = \frac{1}{3}dx \Rightarrow dx = 3du$

$x = 3 \Rightarrow u = 3$ and $x = 15 \Rightarrow u = 7$

Thus $\int_3^{15} f\left(\frac{1}{3}x + 2\right) dx = \int_3^7 3f(u)\, du$

$$= 3\int_3^7 f(u)\, du = 3(11) = 33$$

11. (4-points) Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{2+3x^2}{\sqrt{1-4x-2x^3}}$, $y(-2) = 0$.

$$y = \int \frac{2+3x^2}{\sqrt{1-4x-2x^3}}\, dx$$

Put $u = \sqrt{1-4x-2x^3} \Rightarrow du = \dfrac{-4-6x^2}{2\sqrt{1-4x-2x^3}}\, dx$

$$\Rightarrow du = \frac{-(2+3x^2)}{\sqrt{1-4x-2x^3}}\, dx$$

$$\Rightarrow y = \int -du = -u + C$$

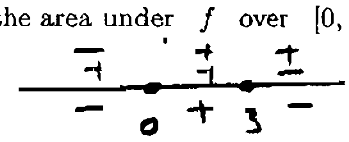
$$\Rightarrow y = -\sqrt{1-4x-2x^3} + C$$

$$y(-2) = 0 \Rightarrow 0 = -\sqrt{25} + C \Rightarrow C = 5$$

$$\Rightarrow y = -\sqrt{1-4x-2x^3} + 5$$

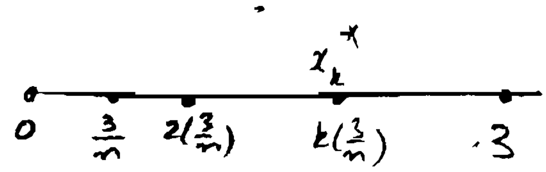
12. (4-points) Find the Riemann sum $R_n$ for $f(x) = 6x - 2x^2$ over the interval $[0,3]$, where the interval is partitioned into $n$ equal subintervals and the right endpoint of each subinterval is used. Then use limits to find the area under $f$ over $[0,3]$.

$$f(x) = 6x - 2x^2 = 2x(3-x)$$



$$\Rightarrow f(x) \geq 0 \text{ on } [0,3]$$

$$x_k^* = k\left(\frac{3}{n}\right) = \frac{3k}{n}, \quad \Delta x = \frac{3}{n}$$



$$\Rightarrow R_n = \sum_{k=1}^{n} f(x_k^*)\, \Delta x$$

$$= \sum_{k=1}^{n} \left[6\left(\frac{3k}{n}\right) - 2\left(\frac{3k}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \frac{54}{n^2} \sum_{k=1}^{n} k - \frac{54}{n^3} \sum_{k=1}^{n} k^3$$

$$= \frac{54}{n^2} \frac{n(n+1)}{2} - \frac{54}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$\text{Area} = \lim_{n \to \infty} R_n = \frac{54}{2} - \frac{(54)(2)}{6} = 27 - 18 = 9$$