MATH 260 Exam II (072)

1. (4-points) Express the matrix $A = \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$ as a product of elementary matrices.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 6 & 1 & 0 \end{array}\right] \qquad E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & \boxed{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right] \qquad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

2. (6-points) Find the particular solution to the DE $y'' - 6y' + 25y = 0$ which satisfies the conditions $y(0) = -2$ and $y\left(\frac{\pi}{8}\right) = e^{\frac{3\pi}{8}}$.

The characteristic equation is

$$r^2 - 6r + 25 = 0 \Rightarrow r = \frac{6 \pm \sqrt{36-100}}{2}$$

$\Rightarrow r = 3 \pm 4i \Rightarrow$ The G.S. of the DE is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$y(0) = -2 \Rightarrow \boxed{-2 = c_1}$$

$$y\left(\frac{\pi}{8}\right) = e^{3\pi/8} \Rightarrow e^{3\pi/8} = e^{3\pi/8}(0 + c_2)$$

$\Rightarrow \boxed{c_2 = 1}$. Thus the required particular solution is $y = e^{3x}(-2\cos 4x + \sin 4x)$.

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3. (4-points) Use Cramer's Rule to find the value of y given in the system

$$\begin{aligned} x+4y+5z&=1\\ 4x+2y+5z&=-1\\ -3x+3y-z&=2 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 4 & 5\\ 4 & 2 & 5\\ -3 & 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x\\ y\\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1\\ -1\\ 2 \end{bmatrix}}_{B}$$

$$\Rightarrow |A| = \left|\begin{array}{ccc|cc} 1 & 4 & 5 & 1 & 4\\ 4 & 2 & 5 & 4 & 2\\ -3 & 3 & -1 & -3 & 3 \end{array}\right| = (-2-60+60)-(-30+15-16)$$

$$= -2-(-31) = 29$$

$$\& \quad |A_y| = \left|\begin{array}{ccc|cc} 1 & 1 & 5 & 1 & 1\\ 4 & -1 & 5 & 4 & -1\\ -3 & 2 & -1 & -3 & 2 \end{array}\right| = (1-15+40)-(15+10-4)$$

$$= 26-21 = 5$$

$$\Rightarrow y = \frac{|A_y|}{|A|} = \frac{5}{29}.$$

4. (4-points) Find the matrix adj A where $A = \begin{bmatrix} -1 & 2 & 0\\ 0 & 3 & 1\\ 2 & 0 & -2 \end{bmatrix}$.

$$\text{Cof } A = \begin{bmatrix} -6 & 2 & -6\\ 4 & 2 & 4\\ 2 & 1 & -3 \end{bmatrix}$$

$$\Rightarrow \text{adj } A = (\text{Cof } A)^T = \begin{bmatrix} -6 & 4 & 2\\ 2 & 2 & 1\\ -6 & 4 & -3 \end{bmatrix}$$

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5. (4-points) Find the general solution of the DE $2y''' - 17y'' + 48y' - 45y = 0$, given that e^{3x} is one of its solutions.

$\Rightarrow$ 3 is a root of the characteristic equation

$$2r^3 - 17r^2 + 48r - 45 = 0$$

Use synthetic division

$$\begin{array}{r|rrrr} 3 & 2 & -17 & 48 & -45 \\ & & 6 & -33 & 45 \\ \hline 3 & 2 & -11 & 15 & 0 \\ & & 6 & -15 & \\ \hline & 2 & -5 & 0 & \end{array}$$

$\Rightarrow$

$$(r-3)^2(2r-5) = 0$$

$\Rightarrow r = 3, 3, 5/2 \Rightarrow$

The G.S. is

$$y = (c_1 + c_2 x)e^{3x} + c_3 e^{\frac{5}{2}x}$$

6. (3-points) Use the Wronskian to determine whether the set $\{1+x, 2x-3x^2, 15x^2\}$ is linearly independent or linearly dependent on the interval $(-0, \infty)$.

$-\infty$

$$W = \begin{vmatrix} 1+x & 2x-3x^2 & 15x^2 \\ 1 & 2-6x & 30x \\ 0 & -6 & 30 \end{vmatrix}$$

$$= (1+x)\left[60 - \cancel{180x} + \cancel{180x}\right] - \left[60x - \cancel{180x^2} + \cancel{180x^2}\right]$$

$$= 60 + 60x - 60x = 60 \neq 0 \text{ on } (-\infty, \infty)$$

$\Rightarrow$ The given set is linearly independent.

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7. (4-points) Determine whether or not the vector $\mathbf{u} = (5, -6, -8)$ is a member of the subspace of R^3 spanned by the vectors $\mathbf{v} = (1, 3, 5)$ and $\mathbf{w} = (-1, 4, 6)$.

Let $u = \alpha v + \beta w \Longrightarrow$

$(5, -6, -8) = \alpha(1, 3, 5) + \beta(-1, 4, 6)$

$$\Longrightarrow \left.\begin{array}{l} \alpha - \beta = 5 \\ 3\alpha + 4\beta = -6 \end{array}\right\} \Longrightarrow 7\beta = -21 \Rightarrow \beta = -3 \Rightarrow \alpha = 2$$

$5\alpha + 6\beta = -8$

Substitute the values $\alpha = 2$, $\beta = -3$ in the third eqn $\Longrightarrow$ $LHS = 10 - 18 = -8 = RHS$.

$\Longrightarrow (5, -6, -8) = 2(1, 3, 5) - 3(-1, 4, 6)$

$\Longrightarrow u$ is a member of $\text{span}\{v, w\}$.

8. (3-points) Let A be a 3×3 nonsingular matrix with $|A| = 2$. Find the determinant $|\text{adj}\, A|$.

$|A| = 2 \Longrightarrow A$ is nonsingular $\Longrightarrow$.

$A^{-1} = \frac{1}{|A|} \text{adj}\, A \Longrightarrow |\text{adj}\, A| = \big|\, |A|\, A^{-1} \big|$

$\Longrightarrow |\text{adj}\, A| = |A|^3\, |A^{-1}| = |A|^3 \cdot \frac{1}{|A|} = |A|^2$

$= 4.$

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9. (6-points) Find a basis for and the dimension of the solution space of the system

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 - 3x_2 + 8x_3 - 13x_4 &= 0 \\ 7x_1 - 3x_2 + 13x_3 - 23x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -3 & 8 & -13 & 0 \\ 7 & -3 & 13 & -23 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -5 & 10 & -15 & 0 \\ 0 & -16 & 20 & -30 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -5 & 10 & -15 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & \boxed{1} & -1 & 1 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = \alpha$ and $x_4 = \beta \Longrightarrow$

$x_1 = -\alpha + 2\beta$, $x_2 = 2\alpha - 3\beta$, $x_3 = \alpha$, $x_4 = \beta$

and any solution is of the form

$(-\alpha + 2\beta, 2\alpha - 3\beta, \alpha, \beta) = \alpha(-1, 2, 1, 0) + \beta(2, -3, 0, 1)$

$\Longrightarrow$ The solution space $=$ span $\{(-1, 2, 1, 0), (2, -3, 0, 1)\}$

If $\alpha(-1, 2, 1, 0) + \beta(2, -3, 0, 1) = (0, 0, 0, 0)$

$\Longrightarrow \alpha = 0$ and $\beta = 0 \Longrightarrow$

A basis is $\{(-1, 2, 1, 0), (2, -3, 0, 1)\}$

The dimension $= 2$

Bonus Problems MATH 260 EXAM II

(I) (5-points) Show that if the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then the set $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is also linearly independent.

Let $\alpha u + \beta(u+v) + \gamma(u+v+w) = 0$ ①

We need to show that $\alpha = \beta = \gamma = 0$

Now ① $\Rightarrow (\alpha+\beta+\gamma)u + (\beta+\gamma)v + \gamma w = 0$ ②

But $\{u, v, w\}$ is l.i. Thus ② $\Rightarrow$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta+\gamma &= 0 \\ \gamma &= 0 \end{aligned}\right\}$$ use back substitution $\Rightarrow$ $\gamma = 0 \Rightarrow \beta = 0 \Rightarrow \alpha = 0$ i.e $\alpha = \beta = \gamma = 0$.

$\Rightarrow \{u, u+v, u+v+w\}$ is l.i

(II) (5-points) Show that the solution set of the homogeneous linear second-order DE $A(x)y'' + B(x)y' + C(x)y = 0$ is a subspace of the vector space of all differentiable functions.

Let y_1 and y_2 be any two solutions $\Rightarrow$

$$LHS = A(x)[y_1+y_2]'' + B(x)[y_1+y_2]' + C(x)[y_1+y_2]$$
$$= \left(A(x)y_1'' + B(x)y_1' + C(x)y_1\right)$$
$$+ \left(A(x)y_2'' + B(x)y_2' + C(x)y_2\right)$$
$$= 0 + 0 = 0 \Rightarrow y_1+y_2 \text{ is a solution}$$

Now let α be any nonzero scalar $\Rightarrow$

$$A(x)(\alpha y_1)'' + B(x)(\alpha y_1') + C(x)(\alpha y_1)$$
$$= \alpha\,[A(x)y_1'' + B(x)y_1' + C(x)y_1] = \alpha \cdot 0 = 0.$$

$\Rightarrow \alpha y_1$ is a solution $\Rightarrow$ The solution space