7.1, 7.2, 7.3, 7.5 Linear Systems of DEs

7.1 First-Order Systems

Objectives 01 To transform a given DE into an equivalent system of first-order DEs.

02 To solve a system by Elimination.

01 Examples: ① Replace the DE $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - x = e^t$ by a system of first order DE

Solution Let $x_1 = x$, $x_2 = x'$ $\Rightarrow$

$x_1' = x_2$ and $x'' = -2x' + x + e^t \Rightarrow$

$x_2' = -2x_2 + x_1 + e^t \Rightarrow$ The required system is

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 - 2x_2 + e^t \end{cases} \quad \text{(I)}$$

Note: In this chapter "t" will be the independent variable.

The matrix form of (I) is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

---

② Transform $x''' - 4x'' + 2x' - 5x = 0$ into a system of first-order DEs.

Solution: Let $x_1 = x$, $x_2 = x'$, $x_3 = x'' \Rightarrow$

$$x''' = 5x - 2x' + 4x'' \Rightarrow$$

$$x_3' = 5x_1 - 2x_2 + 4x_3 \Rightarrow$$ The required system is:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 5x_1 - 2x_2 + 4x_3 \end{cases} \quad \text{II}$$

II in Matrix form is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{02}$ Examples Use the elimination method to solve the following systems

Example 1 $\dfrac{dx_1}{dt} = x_2$ (i)

$\dfrac{dx_2}{dt} = -2x_1 + 3x_2$ (ii)

solution (i) $\Rightarrow \dfrac{d^2x_1}{dt^2} = \dfrac{dx_2}{dt_2}$ (iii)

(i), (ii), (iii) $\Rightarrow \dfrac{d^2x_1}{dt^2} = -2x_1 + 3\dfrac{dx_1}{dt}$ (iv)

$\Rightarrow \dfrac{d^2x_1}{dt^2} - 3\dfrac{dx_1}{dt} + 2x_1 = 0$. Solve for x_1

$r^2 - 3r + 2 = (r-1)(r-2) = 0 \Rightarrow r = 1, 2 \Rightarrow$

$$x_1 = c_1 e^{t} + c_2 e^{2t} \Longrightarrow$$

$$x_2 = \frac{dx_1}{dt} = c_1 e^{t} + 2c_2 e^{2t}.$$

Thus, the G.S. of the given system is:

$$x_1 = c_1 e^{t} + c_2 e^{2t}$$

$$x_2 = c_1 e^{t} + 2c_2 e^{2t}$$

Example ② Solve the intial-value problem

$$\begin{cases} x' = -y \\ y' = 13x + 4y \end{cases} \quad ; \; x(0) = 0, \; y(0) = 3$$

Solution: $x'' = -y' = -(13x + 4y) \Longrightarrow$

$x'' = -13x - 4x' \Longrightarrow x'' + 4x' + 13x = 0$

$\Longrightarrow r^2 + 4r + 13 = 0 \Longrightarrow r = \dfrac{-4 \pm \sqrt{16 - 52}}{2} \Longrightarrow$

$r = -2 \pm 3i \Longrightarrow x = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$

$\Longrightarrow y = -x' = e^{-2t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$

$\qquad - 2e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$

$\Longrightarrow$ The G.S. is $x = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$

& $y = e^{-2t}\left[-(3c_1 + 2c_2)\sin 3t + (3c_2 - 2c_1)\cos 3t\right]$

Use $x(0) = 0$ & $y(0) = 3 \Longrightarrow$

$\boxed{0 = c_1}$ & $3 = 3c_2 \Rightarrow c_2 = 1$

$\Rightarrow$ The solution of the given IVP is

$$x = e^{-2t} \sin 3t$$

$$y = e^{-2t}(2 \sin 3t + 3 \cos 3t)$$

$\boxed{7.2}$ Matrices And Linear Systems

Objectives $\boxed{01}$ Derivatives of Matrices

$\boxed{02}$ Solution vectors and Linear Independence of solutions

$\boxed{01}$ • If $A = [f_{ij}(t)] \Rightarrow A' = [f'_{ij}(t)]$.

• If $A = [f_{ij}(t)]$ and $B = [g_{ij}(t)]$ are $n \times n$ matrices $\Rightarrow (AB)' = A'B + AB'$.

Example ③ If $A = \begin{bmatrix} t & 4t+3 \\ t^2+1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2t \\ t^3 & 3t \end{bmatrix}$

verify that $(AB)' = A'B + AB'$. [Exercise].

$\boxed{02}$ Example ③ Verify that the vector

$X_1 = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} e^{4t}$ is a solution of the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>Solution</u> $LHS = \frac{d}{dt}\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} e^{4t} = 4\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} e^{4t}$

$$RHS = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} e^{4t} = \begin{bmatrix} 8 \\ -12 \\ -12 \end{bmatrix} e^{4t} = 4\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} e^{4t}$$

$\Rightarrow LHS = RHS \Rightarrow$ The given vector is a solution.

<u>Exercise</u>: verify that $X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t}$ is also a solution of the system in example 2.

$\boxed{02}$ Let $X_1, X_2, \dots, X_n$ be n solutions of the system $\frac{d}{dt}X = AX$ where A is an $n \times n$ matrix.

(I) The set $\{X_1, X_2, \dots, X_n\}$ is linearly independent if the determinant $|X_1\ X_2 \cdots X_n| \neq 0$.
This determinant is called the Wronskian of $X_1, \dots, X_n$ and will be denoted by $W(X_1, \dots, X_n)$.

(II) If $\{X_1, X_2, \dots, X_n\}$ is a set of linearly independent solution vectors of the system $\frac{d}{dt}X = AX$, where A is $n \times n$, then the vector $X = c_1X_1 + c_2X_2 + \dots + c_nX_n$ is the general solution of the given system.

Example 4 Show that

$$X = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{5t}$$

is the general solution of the system

$$X' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} X .$$

Solution (Exercise) Hint: Show that

$$X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{t}, \quad X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{3t}, \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{5t}$$

are solutions of the given system, then show that $W(X_1, X_2, X_3) \neq 0$ for all t

---

7-3 The Eigenvalue Method for Linear Systems

Important Result: Let v be an eigenvector associated with the eigenvalue λ of the matrix A. Show that $X = v e^{\lambda t}$ is a nontrivial solution of the system $X' = AX$

Solution: LHS $X' = (v e^{\lambda t})' = \lambda v e^{\lambda t}$

$$RHS = AX = A(v e^{\lambda t}) = (Av) e^{\lambda t}$$
$$= (\lambda v) e^{\lambda t} = \lambda v e^{\lambda t} = LHS.$$

$\Rightarrow X = v e^{\lambda t}$ is a solution of $X' = AX$.

Example 5 Distinct Real Eigenvalues

Solve the $X' = AX$, where $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$.

Solution We find the eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 9\right] = 0$$

$\Rightarrow \lambda = 1$ or $\lambda^2 - 2\lambda - 8 = 0 = (\lambda-4)(\lambda+2) = 0$

The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -2$

Then we find an associated eigenvector for each eigenvalue:

$\boxed{\lambda_1 = 1}$ $\left[\begin{array}{ccc|c} 0 & -1 & \boxed{-1} & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right] \longrightarrow$

$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = \text{free}, \; x_2 = x_3 = 0 \Rightarrow$

an associated eigenvector to $\lambda_1 = 1$ is $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$ is a solution of the system

$\boxed{\lambda_2 = 4}$ $\left[\begin{array}{ccc|c} -3 & -1 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & \boxed{3} & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -3 & \boxed{-1} & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} -3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_3 = \text{free} = \alpha, \; x_2 = \alpha, \; x_1 = -\frac{2}{3}\alpha$

let $\alpha = 3 \Longrightarrow v = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$ is an eigenvector

associated to $\lambda_2 = 4 \Rightarrow X_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} e^{4t}$

is a solution of the system.

$\boxed{\lambda_3 = -2}$ $\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & \boxed{3} & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & \boxed{-1} & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow x_3 = \text{free} = \alpha,\ x_2 = -\alpha,\ x_1 = 0$

Let $\alpha = 1 \Rightarrow v = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t}$

is a solution of the system.

Since the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = -2$ are distinct, then the associated eigenvectors X_1, X_2, and X_3 as obtained above are linearly dependent [You may use the Wronskian to confirm that] $\Rightarrow$ The G.S. of the system is $X = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} e^{4t} + c_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t}$.

Complex Eigenvalues

As you have seen before: If $z = x + iy \Rightarrow$ $\bar{z} = x - iy$ where $i = \sqrt{-1}$ and x, y are real $\Rightarrow$

$$\left\{\begin{array}{l} \frac{1}{2}(z + \bar{z}) = x = \text{real part of } z \text{ and} \\ \frac{i}{2}(-z + \bar{z}) = y = \text{imaginary part of } z \end{array}\right\} \quad (i)$$

Now let v be an eigenvector associated with the complex eigenvalue $\lambda_1 = \alpha + i\beta$ of the matrix A, then $\bar{v}$ is an eigenvector associated with the eigenvalue $\lambda_2 = \bar{\lambda}_1 = \alpha - i\beta$ of A.

$\Rightarrow Y_1 = v e^{\lambda_1 t}$ and $Y_2 = \bar{Y}_1 = \bar{v} e^{\bar{\lambda}_1 t}$ are two linearly independent solutions of the system

$$X' = AX$$

$\Rightarrow X_1 = \frac{1}{2}(Y_1 + \bar{Y}_1)$ and $X_2 = \frac{i}{2}(-Y_1 + \bar{Y}_1)$ are also two l.i solutions of $X' = AX$.

Now use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ and $\frac{1}{2}(z + \bar{z})$ = real part of z and $\frac{1}{2}(z - \bar{z})$ = Imaginary of z $\Longrightarrow$

$$X_1 = e^{\alpha t}\left[B_1 \cos\beta t - B_2 \sin\beta t\right]$$

$$\& \quad X_2 = e^{\alpha t}\left[B_2 \cos\beta t + B_1 \sin\beta t\right]$$

where B_1 = Real part of v and

B_2 = Imaginary part of v

The form of the solutions associated with the complex eigenvalue $\lambda = \alpha + i\beta$

The following Example is an application of the above result:

Example 6 (Complex Eigenvalues)

Solve the system $X' = AX$, where $A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$

Solution: The characteric equation $\begin{vmatrix} 2-\lambda & 1 \\ -4 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8 = 0$

$\Rightarrow \lambda_1 = 2+2i$ and $\lambda_2 = \bar{\lambda}_1 = 2-2i \Rightarrow$ An eigenvector associated with $\lambda_1 = 2+2i$: $\left[\begin{array}{cc|c} -2i & 1 & 0 \\ \boxed{-4} & -2i & 0 \end{array}\right] \xrightarrow{2iR_1+R_2}$

$\left[\begin{array}{cc|c} -2i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2ix_1 + x_2 = 0 \Rightarrow x_2 = 2ix_1$. Let

$x_1 = 1 \Rightarrow x_2 = 2i \Rightarrow v = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 2+2i \Rightarrow$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \text{Real } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and

$\text{Imag } v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow$ Two linearly independent solutions of the system are (see page 9)

$$X_1 = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin 2t \right)$$

$$X_2 = e^{2t}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right)$$

The G.S. is $X = c_1 X_1 + c_2 X_2$.

7.5 Multiple Eigenvalue Solutions

Case I : Complete Eigenvalues

Example 7: Solve the system $X' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X$.

Solution The characteristic equation:

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \overset{(\lambda-1)(\lambda+1)}{(\lambda^2-1)} + (\lambda+1) + (\lambda+1)$$

$$= -(\lambda+1)\left[\lambda^2 - \lambda - 2\right] = -(\lambda+1)^2(\lambda-2) = 0$$

$\Rightarrow$ The eigenvalues are : $\lambda = -1$ (of multiplicty 2), i.e $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 2$. The question now: Can we get three linearly independent eigenvectors associated with the eigenvalues $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$ where one of them is <u>repeated</u>? We solve and see:

$\boxed{\lambda_1 = \lambda_2 = -1}$ solve $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 = -x_2 - x_3$, $x_2 = \text{free} = \alpha$, $x_3 = \text{free} = \beta$

Thus any vector is of the form $\begin{bmatrix} -\alpha-\beta \\ \alpha \\ \beta \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} -\alpha-\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{\underline{\text{two}}}\ \text{l.i}$$

eigenvectors $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

associated with the <u>repeated</u> eigenvalue $-1 \Rightarrow$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} \quad \text{and} \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

are two <u>l.i</u> solutions of the given system.

For $\boxed{\lambda_3 = 2}$ Solve $\left[\begin{array}{ccc|c} \boxed{-2} & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 1 & \boxed{1} & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_3 = \text{free} = \alpha$$

and $x_1 = \alpha$, $x_2 = \alpha$. Let $\alpha = 1 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is an eigenvector associated with the eigenvalue 2.

$\Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$ is a solution of the system

and $\{X_1, X_2, X_3\}$ is a l.i set of solutions

$\Rightarrow$ Enough <u>l.i</u> solution $\Rightarrow$ The G.S. is

$$X = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}.$$

<u>Complete Eigenvalue</u>: An eigenvalue of multiplicity k is called <u>complete</u> if it has k linearly independent associated eigenvectors.

Thus the eigenvalue -1 in Example 7 above incomplete since it is of multiplicity 2 and it has two linearly independent eigenvectors associated with it.

Case II Defective Eigenvalues

An eigenvalue λ of multiplicity k is called **defective** if it is **Not complete**.

In this section we study the case of "**Defective Multiplicity 2 Eigenvalues**"

Algorithm: Let v_1 be an eigenvector associated with a defective multiplicity 2 eigenvalue λ of a matrix A $\Longrightarrow$ $X_1 = v_1 e^{\lambda t}$ is a solution of the system $X' = AX$.

To find another solution X_2 of the given system such that $\{X_1, X_2\}$ is l.i:

we find a vector v_2 such t

$$(A - \lambda I)^2 v_2 = 0$$

and $(A - \lambda I) v_2 = v_1$

$$\Longrightarrow \quad X_2 = (v_1 t + v_2) e^{\lambda t}.$$

Example 8: Solve $X' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} X$.

Solution: Eigenvalues: $\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \Longrightarrow$

$\lambda = 2$ of multiplicity 2.

Associated eigenvectors: $\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a generator for the eigenspace $E_2 \Rightarrow \dim E_2 = 1 \Rightarrow \lambda = 2$ is a defective multiplicity 2 eigenvalue.

We follow the above algorithm to find v_2 which satisfies $(A - \lambda I)^2 v_2 = 0$ and $(A - \lambda I) v_2 = v_1$.

$$(A - \lambda I)^2 v_2 = 0 \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ could be any vector $\in R^2$. So we find v_2 such that $(A - 2I) v_2 = v_1 \Rightarrow$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow 3\beta = 1 \text{ and } \alpha = \text{free}$$

Let $\alpha = 0 \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$.

Now let $X_2 = (v_1 t + v_2) e^{\lambda t} \Rightarrow$

$$X_2 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \right) e^{2t} = \begin{bmatrix} t \\ 1/3 \end{bmatrix} e^{2t}$$

$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$ and $X_2 = \begin{bmatrix} t \\ 1/3 \end{bmatrix} e^{2t}$ are two linearly independent solutions of the given system $\Rightarrow$ G.S: $X = c_1 X_1 + c_2 X_2$.