Iterative Methods for Solving Variational Inequalities with Applications

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TABLE OF CONTENTS

ملخّص

في هذا المشروعِ، سوف نَدْرسُ طريقةَ مهجنة للطريقة النزول الحادة وكذلك طريقة التقريب اللزجة بطريقة أعم ولكن مع شروطٍ أقل من الشروط الموجودة في البحوث السابقة. بشكل خاص، نحن سَنَقترحُ ونُحلّلُ طريقة التقريب التكرارية اللزجة المخففة لإيجاد نقطة ثابتة مشتركة لعائلة أبدالية غير مكلفة بإسقاطها على مجموعة محدبة مغلقة C لفضاء باناخ المخفف. وسوف نثبت أن متتالية الحلول المقربة المولدة بطريقة المقترحة تتقارب بقوة إلى حل متباينة التغير. إن طريقتنا التقريبية التكرارية اللزجة المخففة هي عبارة عن امتداد مختلف عن طريقة التقريب التكرارية اللزجة الأصلية. إن النتائج المتوقعة تعتبر تحسين مهم وتعميم لنتائج الموجودة في البحوث [17,23,28,25,38,43] وغيرها. سوف نقترح طريقةَ مهجنة للطريقة النزول الحادة وكذلك طريقة التقريب اللزجة لمتباينة التغير العامة. وحيث أنه لا يوجد طريقة تقريب شبيه بطريقة المذكورة لحساب النقطة الثابتة لإسقاط متعدد القيم، لذلك فإن تقديم طريقةَ مهجنة للطريقة النزول الحادة وطريقة التقريب اللزجة لمتباينة التغير العامة سوف تكون مهمة صعبة. سوف نطبق هذه الطريقه على كثير من المسائل الرياضية وسوف نقوم ببناء برامج لحل هذه المسائل.

Abstract

In this project, we shall study the hybrid steepest descent method and viscosity approximate method in a more general setting with some mild conditions than those given in the literature. In particular, we shall suggest and analyze a relaxed viscosity iterative method for finding a common fixed point of a commutative family of nonexpansive self-mappings on a closed convex set of a reflexive Banach space. We shall also prove that the sequence of approximate solutions generated by the proposed method converges strongly to a solution of a variational inequality. Our relaxed viscosity iterative method would be an extension and a variant form of the original viscosity iterative method. Our results could be viewed as significant improvement and generalization of the corresponding results in Halpern [17], Lions [23], Reich [28], Moudafi [25], Xu [38], Yao and Noor [43] and many others.

We shall propose hybrid steepest descent method and viscosity approximate method for a general variational inequality. Since no approximate method, similar to the above mentioned method, is available for computing the fixed points of a multivalued map; it is really a very difficult task to put forward hybrid steepest descent method and viscosity approximate method for generalized variational inequalities. We shall try to achieve this goal. As applications of our new methods, we shall solve the pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem. We shall also write computer programs for our methods and shall demonstrate their applications for a pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

Key Words

Hybrid steepest descent method, Viscosity approximate method, Fixed points, Variational inequalities, General variational inequalities, Pseudoinverses, Convex / Quadratic optimization, Semi-definite programming problem.

1. Introduction

The theory of variational inequalities and fixed point theory are two important and dynamic areas in nonlinear analysis and optimization. The theory of variational inequalities introduced by Stampacchia [32] in the early sixties has played a vital role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, operations research, game theory, mathematical economics and engineering sciences; See, for example, [1, 6, 9, 12, 13, 16, 18, 20, 21, 22, 27, 46] and references therein. It has been extended and generalized in various directions using innovative techniques. In the last three decades, many researchers studied the following three aspects of the variational inequalities and their generalizations.

- **Existence Theory:** The study of the existence of solutions of variational inequalities and their generalizations.
- **Numerical Methods:** The study of the algorithms for computing the approximate solutions of variational inequalities and their generalizations and the study of the convergence of the approximate solutions obtained by the algorithm to the exact solution of the problem.
- **Applications:** The study of problems from science, social sciences, engineering, etc by using variational inequality technique.

Several methods are used in the literature to study the existence of solutions and to develop some algorithms for computing the approximate solutions of variational inequalities and their generalizations. The projection method is one of the best and elegant an one. Due to the complexity of the convex set involved in the formulation of variational inequalities, some time it may not be easy to compute the projection of a convex set. Keeping this view point in mind, Yamada [40] (see also [7]) replaced the convex set involved in the formulation of variational inequalities by a set of fixed points of a nonexpensive self map and then introduced a method for studying the existence of solutions and for computing the approximate solutions of variational inequalities, called hybrid steepest descent method. It is also studied by replacing the set of fixed points of a nonexpensive self map with the set of common fixed points of finite number of nonexpensive commutative maps. As another application, Yamada [40] applied his method to study the constrained pseudoinverse problems. This method is further studied by Xu and Kim [39] and Zeng et al [47].

Closely related to variational inequalities and optimization theory, is the problem of finding a common fixed point of a given family of operators. In this direction, several iterative methods have been developed for these problems, see for example [15, 34] and references therein. In recent years, viscosity approximation methods have been developed for finding the approximate solutions of the family of operators. In 2000, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [25]. He also established the strong convergence of both the implicit and explicit iterative schemes in Hilbert spaces. Under certain conditions, it is proved that the limit point of the iterative sequence is unique (common) fixed point of a self nonexpensive map (finite family of self nonexpensive maps) and this limit point is also a solution of a variational inequality. Subsequently, Xu [38] extended Moudafi's results in the framework of Hilbert spaces and proved the strong convergence of the continuous scheme and the iterative scheme in a uniformly smooth Banach space. Very recently, Yao and Noor [43] considered and analyzed a new viscosity iterative method for finding the common fixed point of a family of operators in reflexive Banach spaces. Also, they proved that the approximate solution converges strongly to a solution of a variational inequality under some mild conditions.

In this project, we shall study the hybrid steepest descent method and viscosity approximate method in a more general setting under some weaker conditions than those considered in the literature. In particular, we shall suggest and analyze a relaxed viscosity iterative method for finding a common fixed point of a commutative family of nonexpansive self-mappings on a closed convex subset of a reflexive Banach space. We shall also prove that the sequence of approximate solutions generated by the proposed method converges strongly to a solution of a variational inequality. Our relaxed viscosity iterative method would be an extension and a variant of the original viscosity iterative method. Our results could be viewed as significant improvement and generalization of the corresponding results in Halpern [17], Lions [23], Reich [28], Moudafi [25], Xu [38], Yao and Noor [43] and many others. We shall propose hybrid steepest descent method and viscosity approximate method for a general variational inequality. Since no approximate method, similar to above mentioned method, is available for computing the fixed points of a multivalued map, it is really a very difficult task to present hybrid steepest descent method and viscosity approximate method for generalized variational inequalities. We intend to find a solution of this basic problem. As applications of our new methods, we shall solve the pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem. We shall also write computer programs for our methods and demonstrate their relevences to the pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

2. Literature Review and Mathematical Formulations

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H, and $F : H \to H$ be an operator. Stampacchia [32] initially studied the classical variational inequality problem VI(F, C): find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \text{for all } y \in C.$$

Variational inequalities have been extensively studied because they cover many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, etc as special cases; See, for example, [1, 6, 9, 12, 13, 16, 18, 20, 21, 22, 27, 46] and the references therein.

Let $g : H \to H$ be a nonlinear mapping. A generalized form of the variational inequality problem is the following general variational inequality problem VI(F, C, g): find $x^* \in H$ such that $g(x^*) \in C$ and

$$\langle F(x^*), g(y) - g(x^*) \rangle \geq 0, \quad \text{for all } g(y) \in C.$$

The above inequality is called general variational inequality. The odd-order and non-symmetric free, unilateral, obstacle and equilibrium problems can be studied by using VI(F, C, g); See, for example, [18] and references therein.

When $F : H \to 2^H$ is a multivalued map with nonempty values, then variational inequality problem and general variational inequality problem are called generalized variational inequality problem and general generalized variational inequality problem, respectively. More precisely, let $F : H \to 2^H$ be a multivalued map with nonempty values. The generalized variational inequality problem GVI(F, C) is the following: find $x^* \in C$ and $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in C.$$

It is known that the necessary and sufficient conditions for a point to be a solution of an optimization problem for convex but nondifferentiable functions is that the point be a solution of GVI(F, C). The general generalized variational inequality problem GVI(F, C, g) is the following: find $x^* \in H$ and $u^* \in F(x^*)$ such that $g(x^*) \in C$ such that

$$\langle u^*, g(y) - g(x^*) \rangle \geq 0, \quad \text{for all } g(y) \in C.$$

It is well known that if F is strong monotone and Lipschitzian on C, then $\mathrm{VI}(F, C)$ has a unique solution, see [21]. In the study of the $\mathrm{VI}(F, C)$, one of the most important problems is: how to find a solution of $\mathrm{VI}(F, C)$ if there is any? A great deal of effort has gone into the problem of finding a solution of $\mathrm{VI}(F, C)$; See, for example, [9, 13, 22, 27].

It is also known that the $\mathrm{VI}(F, C)$ is equivalent to the fixed-point equation

$$x^* = P_C(x^* - \mu F(x^*))$$

where P_C is the (nearest point) projection from H onto C; that is, $P_C x = \operatorname{argmin}_{y \in C} \|x - y\|$ for each $x \in H$ and where $\mu > 0$ is an arbitrarily fixed constant. So if F is strongly monotone and Lipschitzian on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of this equation is a contraction. Hence the Banach contraction principle guarantees that the Picard iterates converge in norm to the unique solution of the $\mathrm{VI}(F, C)$. Such a method is called the projection method. It has been widely extended to develop various algorithms for finding solutions of various classes of variational inequalities and complementarity problems; See [1, 9, 13, 18, 20, 21, 27, 46]. It is remarkable that the fixed-point equation involves the projection P_C which may not be easy to compute due to the complexity of the convex set C. $\mathrm{VI}(F, C, g)$, $\mathrm{GVI}(F, C)$, and $\mathrm{GVI}(F, C, g)$ are studied by using the projection method; See, for example, [8, 10, 24, 26, 30, 31, 45] and references therein.

Recently, Yamada [40] (see also [7]) introduced a hybrid steepest-descent method for solving the $\mathrm{VI}(F, C)$ so as to reduce the complexity, probably caused by the projection P_C. His idea is stated as follows. Let C be the fixed point set of a nonexpansive mapping $T : H \to H$; that is, $C = \{x \in H : Tx = x\}$. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$, and let $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$ denote the fixed-point set of T. Let F be η-strongly monotone and κ-Lipschitzian on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ of real numbers in $(0, 1)$ satisfying the following conditions:

(L1) $\lim_{n\to\infty} \lambda_n = 0$,
(L2) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(L3) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} := Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad n \ge 0. \tag{1}$$

Then Yamada [40] proved that $\{n_n\}$ converges strongly to the unique solution of the $\mathrm{VI}(F, C)$. An example of the sequence $\{\lambda_n\}$ which satisfies conditions (L1)-(L3), is given by $\lambda_n = 1/n^\sigma$ where $0 < \sigma < 1$. We note that condition (L3) was first used by Lions [23] to establish the following result.

THEOREM 2.1. [23] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(S) \ne \emptyset$. For a sequence $\{\alpha_n\}$ in $[0, 1]$ and an arbitrary point $u \in C$, starting with another arbitrary initial $x_0 \in C$ define a sequence $\{x_n\}$ in C recursively by the formula:*

$$x_{x+1} := \alpha_n u + (1 - \alpha_n)Sx_n, \quad n \ge 0. \tag{2}$$

If the sequence $\{\alpha_n\}$ of parameters satisfies conditions (L1), (L2) and (L3), then $\{x_n\}$ converges strongly to an element of $\mathrm{Fix}(S)$.

If C is expressed as the intersection of the fixed-point sets of N nonexpansive mappings $T_i : H \to H$ with $N \ge 1$, an integer, Yamada [40] also proposed another algorithm,

$$u_{n+1} := T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n), \quad n \ge 0 \tag{3}$$

where $T_{[k]} := T_{k \bmod N}$ for integer $k \ge 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$; that is, if $k = jN + q$ for some integers $j \ge 0$ and $0 \le q < N$, then $T_{[k]} = T_N$

if $q = 0$ and $T_{[k]} = T_q$ if $1 \le q < N$ where $\mu \in (0, 2\eta/\kappa^2)$ and where the sequence $\{\lambda_n\}$ of parameters satisfies conditions (L1), (L2) and the following (L4):

(L4) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}|$ is convergent.

Under these conditions, Yamada [40] proved the strong convergence of $\{u_n\}$ to the unique solution of the VI(F, C). Note that condition (L4) was first used by Bauschke [2]. In the special case of $N = 1$, Wittmann [35] first introduced condition (L4) and applied (L4) to establish the following theorem.

Theorem 2.2. [35] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(S) \ne \emptyset$. *For a sequence $\{\alpha_n\}$ in* $[0, 1]$ *and an arbitrary point $u \in C$, starting with another arbitrary initial $x_0 \in C$ define a sequence $\{x_n\}$ in C recursively by the formula (2). If the sequence $\{\alpha_n\}$ of parameters satisfies conditions (L1), (L2) and (L4) with $N = 1$, then $\{x_n\}$ converges strongly to an element of* $\mathrm{Fix}(S)$.

In 2003, Xu and Kim [39] further considered and studied the hybrid steepest-descent algorithms (1) and (3). Their major contribution is that the strong convergence of algorithms (1) and (3) holds with condition (L3) replaced by the condition

(L3)$'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$

and with condition (L4) replaced by the condition

(L4)$'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+N})/\lambda_{n+N} = 0$.

It is clear that condition $(L3)'$ is strictly weaker than condition $(L3)$ coupled with conditions $(L1)$ and $(L2)$; moreover, $(L3)'$ includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$ while $(L3)$ does not. It is easy to see that if $\lim_{n\to\infty} \lambda_n/\lambda_{n+N}$ exists, then condition $(L4)$ implies condition $(L4)'$. However in general, conditions $(L4)$ and $(L4)'$ are not comparable: neither of them implies the other (see [37] for details). Furthermore under conditions $(L1)$, $(L2)$, $(L3)'$ and $(L4)'$, they gave the applications of algorithms (1) and (3) to the constrained generalized pseudoinverses.

Yamada and Ogura [41] considered the hybrid steepest descent method for variational inequalities over the fixed point set of certain quasi-nonexpansive mappings. They also gave some applications to convex optimization problem over the fixed point set of a nonlinear mapping.

Very recently, Zeng et al [47] introduced the following relaxed hybrid steepest-descent algorithms (I) and (II) which are mixed iteration processes of (1)-(3) as follows:

Algorithm (I): Let $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following iterative scheme

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n)[Tx_n - \lambda_{n+1}\mu F(Tx_n)], \quad n \ge 0.$$

Algorithm (II): Let $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following iterative scheme

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n)[T_{[n+1]}x_n - \lambda_{n+1}\mu F(T_{[n+1]}x_n)], \quad n \ge 0.$$

Under the assumption that $\{\alpha_n\}$ satisfies conditions $(L1)$, $(L4)$ with $N = 1$ and under the assumption that $\{\lambda_n\}$ satisfies conditions $(L1)$, $(L2)$, $(L3)'$, they proved that the sequence $\{x_n\}$ generated by Algorithm (I) converges in norm to the unique solution x^* of the VI(F, C). On the other hand, under the assumption that $\{\alpha_n\}$ satisfies conditions $(L1)$, $(L4)$ and under the assumption that $\{\lambda_n\}$ satisfies conditions $(L1)$, $(L2)$, $(L4)'$, they also proved that the sequence $\{x_n\}$ generated by Algorithm (II) converges in norm to the unique solution x^* of the VI(F, C). Furthermore, they applied these two results to consider the constrained generalized pseudoinverses. Note that whenever the sequence $\{\alpha_n\}$ is a constant sequence $\{0\}$, i.e., $\alpha_n = 0$, $\forall n \ge 0$, Algorithms (I) and (II) immediately reduce to algorithms (1) and (3), respectively.

Another method which describes the relationship between variational inequalities and fixed points is called viscosity approximation method. It is proposed by Moudafi [25] for selecting a particular fixed point of a given nonexpansive mapping defined on Hilbert spaces. If $T : C \to C$ is nonexpansive self mapping and $f : C \to C$ is a contraction mapping, then he proved the following results.

THEOREM 2.3. [25] *The sequence $\{x_n\}$ generated by the (implicit) scheme*

$$x_n = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)$$

converges strongly to the unique solution of the following variational inequality:

$$x^* \in Fix(T) \quad \text{such that} \quad \langle (I-f)x^*, x^* - y\rangle \leq 0, \quad \text{for all } y \in Fix(T),$$

that is, the unique solution of the operator $P_{Fix(T)} \circ f$, where $\{\varepsilon_n\}$ is a sequence of positive numbers tending to zero.

THEOREM 2.4. [25] *With an initial guess $z_0 \in C$, define the sequence $\{z_n\}$ by*

$$z_{n+1} = \frac{1}{1+\varepsilon_n} T z_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(z_n).$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=1}^{\infty} \varepsilon_n = \infty$ and $\lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0$. Then $\{z_n\}$ converges strongly to the unique solution of the following variational inequality:

$$x^* \in Fix(T) \quad \text{such that} \quad \langle (I-f)x^*, x^* - y\rangle \leq 0, \quad \text{for all } y \in Fix(T),$$

that is, the unique solution of the operator $P_{Fix(T)} \circ f$.

Recently, Xu [38] studied the viscosity approximation methods for a nonexpansive mapping defined on a uniform smooth Banach space. It is further studied by Shahzad and Udomene [29] for asymptotically nonexpansive mappings in Banach spaces. Simultaneously, Jung [19] also studied the viscosity approximation method for a family of finite commutative nonexpansive mappings in the setting of Banach spaces.

Very recently, Yao [42] considered the viscosity approximation method for a family of finite noncommutative nonexpansive mappings defined on a Hilbert space. Let $\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_N} \in (0, 1]$, $n \in \mathbb{N}$. Given a finite family $T_1, T_2, \ldots, T_N$ of nonexpansive mappings on H, for each n, define the mappings $U_{n_1}, U_{n_2}, \ldots, U_{n_N}$ by

$$\begin{aligned}
&U_{n_1} = \alpha_{n_1} T_1 + (1-\alpha_{n_1})I,\\
&U_{n_2} = \alpha_{n_2} T_2 U_{n_1} + (1-\alpha_{n_2})I,\\
&\vdots\\
&U_{n_{N-1}} = \alpha_{n_{N-1}} T_{N-1} U_{n_{N-2}} + (1-\alpha_{n_{N-1}})I,\\
&W_n := U_{n_N} = \alpha_{n_N} T_N U_{n_{N-1}} + (1-\alpha_{n_N})I.
\end{aligned}$$

Such a mapping W_n is called the W_n-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_N}$. For the detail of W_n-mapping, we refer to [5] and references therein.

Let f be a contraction on H and choose any $x_0 \in H$. Then, Yao [42] defined the sequence $\{x_n\}$ generated by the following scheme

$$x_{n+1} = \lambda_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \lambda_n A) W_n x_n, \tag{4}$$

where γ, β are two positive real numbers such that $\beta < 1$, W_n is defined as above and A is a self adjoint strongly positive operator.

Yao [42] also proved that the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, y - x^*\rangle \geq 0 \quad \text{for all } y \in \mathscr{F} = \cap_{i=1}^N Fix(T_i),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathscr{F}} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$).

Yao et al [44] extended the results of Yao [42] for an infinite countable family of of nonexpansive mappings in the setting of Banach spaces.

We shall obtain both the methods: hybrid steepest descent method and viscosity approximate method, in a more general setting with some mild conditions than those studied in the literature so far. In particular, we shall suggest and analyze a relaxed viscosity iterative method for finding a common fixed point of a commutative family of nonexpansive self-mappings on a subset on C of a reflexive Banach space. We shall also prove that the sequence of approximate solutions generated by the proposed method converges strongly to a solution of a variational inequality. Our relaxed viscosity iterative method would be an extension and a variant of the original viscosity iterative method. Our results could be viewed as significant improvement and generalization of the corresponding results in Halpern [17], Lions [23], Reich [28], Moudafi [25], Xu [38], Yao and Noor [43] and many others.

We shall try to propose hybrid steepest descent method and viscosity approximate method for a general variational inequality. Since no approximate method, similar to the above mentioned method, is available for computing the fixed points of a multivalued map, it is really a very difficult task to present hybrid steepest descent method and viscosity approximate method for generalized variational inequalities. We will endeavor to talk a positive step in this direction. As applications of our new methods, we shall solve the pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem. We shall also write computer programs for our methods and also shall exhibit their demonstration in the context of pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

3. Objectives of the Study

The main objectives to be achieved are the following:

- We shall establish hybrid steepest descent method and viscosity approximate method in a more general setting under some mild conditions than those given in the literature. In particular, we shall suggest and analyze a relaxed viscosity iterative method for finding a common fixed point of a commutative family of nonexpansive self-mappings on a suitable subset of a reflexive Banach space. We shall also prove that the sequence of approximate solutions generated by the proposed method converges strongly to a solution of a variational inequality. Our relaxed viscosity iterative method would be an extension and a variant of the original viscosity iterative method. Our results could be viewed as significant improvement and generalization of the corresponding results in Halpern [17], Lions [23], Reich [28], Moudafi [25], Xu [38], Yao and Noor [43] and many others.

- We shall propose hybrid steepest descent method and viscosity approximate method for a general variational inequality.

- Since no approximate method, similar to above mentioned method, is available for computing the fixed points of a multivalued map; the difficult task to present hybrid steepest descent method and viscosity approximate method for generalized variational inequalities will be taken up as an integral part of this project.

- As applications of our new methods, we shall solve the pseudoinverse problems and convex / quadratic optimization problems.
- In [11] (see also [14]), the projection method is used to obtain the approximate solutions of semi-definite programming problem. We shall use our new methods to compute the approximate solutions of semi-definite programming problem.
- We shall write the computer programs for our methods and shall demonstrate their applications in pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

4. Methodology / Description

In order to achieve above mentioned objectives, following methodology will be followed:

- **Hybrid Steepest Descent Method and Viscosity Approximate Method:**

  We shall obtain both the methods in a more general setting but with some mild conditions than those already given in the literature. We shall suggest these methods for general variational inequalities and generalized variational inequalities.

- **Applications:**

  we shall use our methods to solve the pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

- **Computer Programming and Demonstrations:**

  We shall write the computer programs for our methods and shall demonstrate their use in solving pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem.

5. Management Plan

All the investigators involved in this research are faculty members in the department of mathematical sciences at KFUPM. In general, the investigators have research interest in nonlinear functional analysis, optimization, and computer programming. The metric fixed point theory is the main research area of the Principle Investigator (Abdul Rahim Khan). The first Co-Investigator (Suliman Al-Homidan) has a very good knowledge of optimization and has a broad spectrum of experience in this area. He is the one who has a good command on semi-definite programming problem and convex / quadratic optimization problem. The second Co-Investigator (Qamrul Hasan Ansari) is one of the researchers who have done a lot of work in the area of variational inequalities and fixed point theory. The third Co-investigator (Shamsuddin Khan) has specialization in the computer programming. Jointly, the team of investigators provides the necessary capabilities to bring the proposed research project to a success and fruitful conclusions.

The Principle Investigator and second Co-investigator will achieve the first three objectives. The first Co-Investigator will work on the applications of our methods to solve pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem. The results to be developed in this project will be more valuable if the corresponding computer algorithms are obtained in their support. The first three investigators have no expertise in developing computer programmes / algorithms. Mr. S. D. Khan (the third Co-PI) is an expert to simulate mathematical problems and their verification by computers through respective algorithms and techniques. Therefore, the presence of Mr. S. D. Khan as a Co-PI is very much needed on regular basis. The third Co-Investigator will write computer programs for our methods and will demonstrate their relevance to finding numerical solutions pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem. In general, the stated objectives would be achieved through the collaborative work of the investigators. The Principle Investigator will be coordinating the report/publication output and perform necessary administrative management of the project.

6. Significance of the Study

- With a mathematical view point, we shall provide refinement of the above mentioned methods in a more general setting under some mild conditions than those already given in the literature. Since no such methods are available for general variational inequalities and generalized variational inequalities, the study of hybrid steepest descent method and viscosity approximate method for these inequalities would provide a new direction. It would also be possible to compute approximate solutions of the odd-order and non-symmetric free boundary value problems, unilateral boundary value problems, and obstacle and equilibrium problems.
- The solution of pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem based on our new methods will present a new direction in the area of investigations.
- The demonstration of our methods for pseudoinverse problem, convex / quadratic optimization problem and semi-definite programming problem through computer programs would be an innovation in the subject of variational inequalities.

7. Work Plan RG-2

TIME TABLE

Task	**Months**																	
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
Literature Collection & Completion of Review																		
Objective # 1 (Hybrid Steepest Descent Method and Viscosity Approximation Method)																		
Objective # 2 (Applications)																		
Objective # 3 (Computer Programming and Demonstrations)																		
Final Report and Publication Write up																		

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9. Proposed Budget

RG-3

COST ESTIMATE

PERFORMANCE PERIOD

FROM: March 01, 2007 or date of approval TO: August 31, 2008

TITLE: ITERATIVE METHODS FOR SOLVING VARIATIONAL INEQUALITIES WITH APPLICATIONS

Principle Investigator: Dr. Abdul Rahim Khan
Co-Investigator: Dr. Suliman Al-Homidan
Co-Investigator: Dr. Qamrul Hasan Ansari
Co-Investigator: Mr. Shamsuddin Khan
Duration of the Project: 18 months
Starting Date: March 01, 2007 or date of approval
Completion Date: August 31, 2008

Total Budget: SR 87,500

Head #	Budget Item	Amount Allocated	Remarks
1.	**Manpower:**	(SR)	
	Dr. Abdul Rahim Khan PI @ SR. 1200 for 18 months	21,600	
	Dr. Suliman Al-Homidan Co-I @ SR. 1000 for 18 months	18,000	
	Dr. Qamrul Hasan Ansari Co-I @ SR. 1000 for 18 months	18,000	
	Mr. Shamsuddin Khan Co-I @ SR. 1000 for 18 months	18,000	
	Secretary	1,500	
	Manpower Sub-Total	**73,500**	
2.	**Misc. Expenses:**		
	Stationary	1,500	
	Allocation for Books	2,500	
	Sub-Total	**4,000**	
3.	**International Conferences:**		
	For PI / Co-PI	10,000	
	Sub-Total	**10,000**	
	TOTAL AMOUNT	**87,500**	

10. Suggested Referees **RG-4**

PROPOSAL / PROJECT TITLE:

Iterative Methods for Solving Variational Inequalities with Applications

PRINCIPAL INVESTIGATOR: Dr. Abdul Rahim Khan

NAMES AND ADDRESSES OF INTERNATIONALLY REPUTABLE REFEREES

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AREAS CLOSELY RELATED TO THE PROJECT/PROPOSAL ARE:
Nonlinear Functional Analysis and Optimization

Name: Abdul Rahim Khan **Signature:**
Date:

TO WHOMSOEVER IT MAY CONCERN

This is to confirm that we submit herewith a research proposal entitled "ITERATIVE METHODS FOR SOLVING VARIATIONAL INEQUALITIES WITH APPLICATIONS" for funding by the University under KFUPM FUNDED RESEARCH GRANTS during the year 2007-2008.

We would like to state that we have not submitted this research proposal, either in part, or in full, or under different title to any funding agencies including KACST, Research Institute, Academic Development Center, or any outside agency and we stand to lose a chance to get financial support or any related action from the University if, at a later date, it is made known that a similar proposal submitted by us to another agency for funding.

We declare that whatever we have stated is true to the best of our knowledge and understanding.

We will inform the Deanship of Scientific Research if we decided to leave KFUPM for more than one academic semester.

Dr. Abdul Rahim Khan
Principle Investigator **Signature**

Dr. Suliman Al-Homidan
Co-Investigator **Signature**

Dr. Qamrul Hasan Ansari
Co-Investigator **Signature**

Mr. Shamsuddin Khan
Co-Investigator **Signature**