ON PRINCIPLE OF EQUICONTINUITY

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Abstract

The main purpose of this paper is to prove some results of uniform boundedness principle type without the use of Baire's category theorem in certain topological vector spaces; this provides an alternate route and important technique to establish certain basic results of functional analysis. As applications, among other results, versions of the Banach-Steinhaus theorem and the Nikodym boundedness theorem are obtained.

1 Introduction

The classical uniform boundedness principle asserts: if a sequence $\{f_n\}$ of continuous linear transformations from a Banach space X into a normed space Y is pointwise bounded, then $\{f_n\}$ is uniformly bounded. The proof of this result is most often based on the Baire's category theorem (e.g. see Theorem 4.7-3 [18] and Theorem 3.17 [26]); the interested reader is referred to Eidelman et al. [10] for a new approach in this context. Several authors have sought proof of this type of results without Baire's theorem in various settings (see, for example, Daneš [4], Khan and Rowlands [16], Nygaard [23] and Swartz [27]).

In 1933, Nikodym [21] proved: If a family M of bounded scalar measures on a σ-algebra $\mathcal{A}$ is setwise bounded, then the family M is uniformly bounded. This result is a striking improvement of the uniform boundedness principle in the space of countably additive measures on $\mathcal{A}$; a Baire category proof of this theorem may be found in ([9], IV.9.8, p. 309). Nikodym theorem has received a great deal of attention and has been generalized in several directions (see, e.g., Darst [5], Drewnowski [8], Labuda [19], Mikusinski [20] and Thomas [28]); in particular, the proofs of this result without category argument for finitely additive measures with values in a Banach space (quasi-normed group) are provided by Diestel and Uhl ([6], Theorem 1, p. 14) and Drewnoski ([7], Theorem 1), respectively. For other related generalizations of this theorem, we refer to the bibliography in [6].

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Recently, Nygaard [22-23] has used the notion of a "thick set" to prove the uniform bounded principle for transformations on a thick subset of a Banach space X with values in another Banach space Y. The concept of a thick set goes back to the ideas of Kadets and Fonf (see [12], [13], [15]). It is worth pointing out that the concept of "thick sets" heavily depends on the dual of X and the development of their theory essentially relies on the Hahn-Banach separation theorem in X. The broader class of "thick sets" contains as a subclass the class of second category sets.

In this paper, certain aspects of the development of the uniform boundedness principle are discussed; in particular, results of the type of uniform boundedness principle are proved on a domain of second category and beyond without employing Baire's category argument. First, we prove a a general principle of equicontinuity for maps on a topological vector space of the second category with values in another topological vector space. A similar result is obtained for transformations on "thick sets" of a complete locally convex space X satisfying the property (N) and taking values in a locally convex space Y; this generalizes the uniform boundedness principle of Nyagaard [23] to a class of locally convex spaces. An analogue of the new result is given for maps from X^* into Y^*. Some versions of the Banach-Steinhaus theorem and the Nikodym boundedness theorem are also given.

2 Notations and Preliminaries

Let P be a family of seminorms on a Hausdorff locally convex space X. Let $B_X = \{x \in X : p(x) \leq 1 \text{ for each } p \in P\}$ and $S_X = \{x \in X : p(x) = 1 \text{ for each } p \in P\}$ (cf. [3], p. III. 13-14). The strong dual X^* of X is a locally convex space (details may be found in [3], p. IV. 14-23). For our purposes, it would be enough to consider the following: Suppose that Ω is a family of bounded subsets of X. The pair $(\Omega, |\cdot|)$ induces a locally convex topology on X^* via the family P^* of seminorms

$$p^*(x^*) = \sup\{|x^*(x)| : x \in A, \quad A \in \Omega\}.$$

Similarly, if Q is a family of seminorms on a locally convex space Y, then Q^* will be the induced family of seminorms defining the locally convex topology on Y^*.

Let X and X^* be in duality. The polar of $A \subset X$ and $B \subset X^*$ are, respectively, defined by

$$A^0 = \{x^* \in X^* : \sup_{x \in A} |x^*(x)| \leq 1\}.$$

$$B^0 = \{x \in X : \sup_{x^* \in B} |x^*(x)| \leq 1\}$$

where we consider X to be embedded in X^{**}, bidual of X (see Yosida [30]).

Locally convex spaces provide a very general framework for the Hahn-Banach theorem and its consequences; in particular, we shall need the following separation result.

Proposition 2.1 ([27], Prop. 13, p. 173). *Let A be a closed and absolutely convex subset of a Hausdorff locally convex space X and $x \notin A$. Then there exists $x^* \in X^*$ such that $|x^*(x)| > 1 \geq \sup\{x^*(y) : y \in A\}$.*

In what follows we will use the terminology of Nygaard [22-23].

A subset A of a normed space X is norming for X^* if for some $\delta > 0$, $\inf_{x^* \in S_{X^*}} \sup_{x \in A} |x^*(x)| \geq \delta$. Analogously, a subset B of X^* is norming for X (or ω^*-norming) if for some $\delta > 0$, $\inf_{x \in S_X} \sup_{x^* \in B} |x^*(x)| \geq \delta$. We say a subset A of X is thin if it is countable union of an increasing sequence of sets which are non-norming for X^*. A set which is not thin, is called a thick set.

The concept of ω^*-thin and ω^*-thick sets can be defined in the same way.

A set A in a complex vector space X is norming if for some $\delta > 0$, $\overline{co}\left(\bigcup_{|r|=1} rA\right) \supset \delta B_X$. However, we shall employ $\overline{co}(\pm A) \supseteq \delta B_X$ for simplicity.

It will be interesting to formulate the above definitions in the context of an arbitrary locally convex space.

Let G be a commutative group. A non-negative valued function q on G is said to be a quasi-norm if it has the following properties for any x, y in G: (i) $q(0) = 0$, (ii) $q(x) = q(-x)$, (iii) $q(x + y) \leq q(x) + q(y)$.

The relationship of Functional Analysis and Measure Theory is not so easy to understand (for some connections, we refer to [14]). Recently, Abrahamsen et al. [1] have established in Prop. 3.2, boundedness of a vector measure by utilizing the concept of a thick set; thereby reflecting growing interaction between these two subjects. Consequently, such an interplay will play a part here.

Let G be a commutative Hausdorff topological group and $\mathcal{R}$ a ring of subsets of a set X. A function $\mu : \mathcal{R} \to G$ is said to be: (i) measure if $\mu(\phi) = 0$ and $\mu(E \cup F) = \mu(E) + \mu(F)$ where E and F are in $\mathcal{R}$ with $E \cap F = \phi$ (ii) exhaustive if for every sequence $\{E_n\}$ of pairwise disjoint sets in $\mathcal{R}$, $\lim_{n\to\infty} \mu(E_n) = 0$.

The notion of a submeasure has been extensively studied by Drewnoski (see [7-8] and the references therein). The applications of this concept are enormous (e.g. see [8] and [24]). Group-valued submeasures have been introduced by Khan and Rowlands [17] and their work has been further investigated by Avallone and Valente [2].

Let G be a commutative lattice group (ℓ-group). A quasi-norm q on G is an ℓ-quasi-norm if $q(x) \leq q(y)$ for all x, y in G with $|x| \leq |y|$ where $|x| = x^+ + x^-$. An ℓ-quasi-norm generates a locally solid group topololgy on G (cf. Proposition 2 2 C [14]). Following Khan and Rowlands [17], a G-valued function μ on $\mathcal{R}$ is a submeasure if $\mu(\phi) = 0$, $\mu(E \cup F) \leq \mu(E) + \mu(F)$ for all E, F in $\mathcal{R}$ with $E \cap F = \phi$ and $\mu(E) \leq \mu(F)$ for all E, F in $\mathcal{R}$ with $E \subseteq F$. Clearly, in this case $\mu(E) \geq 0$ for all E in $\mathcal{R}$.

3 Main Results

Khan and Rowlands [16] have obtained the following improvement of Theorem 2 due to Daneš [4].

Theorem A *([16], Corollary 1). Let X be a topological vector space, $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} x_n = 0$, and $\{p_n\}$ a sequence of real sub-additive functionals on X*

satisfying the condition:
"there exists a sequence $\{a_k\}$ of real numbers, $a_k \to +\infty$ as $k \to \infty$, such that, for each $k, n = 1, 2, \ldots,$ the set $B_{k,n} = \{x \in X : p_n(x) \leq a_k\}$ is closed in X".

If $\limsup_n (\sup_{x \in U} p_n(x)) = +\infty$ for each neighbourhood U of 0 in X, then the set $Z = \{z \in X : \limsup_n p_n(x_n + z) = +\infty \text{ or } \limsup_n p_n(x_n - z) = +\infty\}$ is a residual G_δ-set in X.

The following example reveals that Theorem A is not true, in general, if Z is replaced by either Z^+ or Z^- where $Z^+ = \{x \in X : x > 0\}$, $Z^- = \{x \in X : x < 0\}$ and $Z = Z^+ \cup Z^-$.

Let X be the usual space of real numbers. We assume that $x_n = 0$ for each $n \in \mathbb{N}$.

Define $p_n(x) = n|x| \ \ (x \in X, n \in \mathbb{N})$. Here $Z^- = \phi$ and so Z^- can not be residual G_δ-set while Z^+ is a residual G_δ-set, for $X \setminus Z^+ = \{0\}$ is of first category in X. Thus, either Z^+ or Z^- can be a residual G_δ-set.

As an application of Theorem A, we establish a principle of equicontinuity in the following result; this leads to an alternative proof of the Banach-Steinhaus theorem given by Rudin [25].

Theorem 1 *(Principle of equicontinuity). Let X be a topological vector space of the second category, Y a Hausdorff topological vector space and $\{f_n\}$ a sequence of continuous linear transformations of X into Y such that the set $\{f_n(x)\}$ is bounded for each $x \in X$. Then the sequence $\{f_n\}$ is equicontinuous.*

Proof. Let the topology of Y be determined by a family $\{q_i : i \in I\}$ of $\mathcal{F}$-seminorms (definition and details may be found in [29], p. 1-3). Suppose that the sequence $\{f_n\}$ is not equicontinuous. Then for some continuous quasi-norm q_{i_0}, which for the sake of simplicity we denote by q, and any τ-neighbourhood U of 0 in X, there exist a sequence $\{x_n\}$ in U and a sequence of integers $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$ such that $q(f_{n_k}(x_n)) > k \quad (k = 1, 2, \ldots)$. It follows that $\limsup_n (\sup_{x \in U} q(f_n(x))) = +\infty$. The functionals $q_0 f_n (n = 1, 2, \ldots)$ satisfy the conditions of Theorem A (taking $x_n = 0$ for all $n = 1, 2, \ldots$), and so the set

$$Z = \{z \in X : \limsup_n q(f_n(z)) = +\infty\}$$

is a residual G_δ-set in X. Thus $X \setminus Z$ is of the first category. Since X is of the second category, it follows that Z is non-empty; this implies that there is a point $z_0 \in X$ such that $\limsup_n q(f_n(z_0)) = +\infty$. This contradicts the hypothesis. Thus the sequence $\{f_n\}$ is equicontinuous. ■

An immediate consequence of the above theorem is given below.

Theorem 2 *(Banach-Steinhaus theorem). Let X and Y be as in Theorem 1 and let $\{f_n\}$ be a sequence of continuous linear transformations of X into Y such that $f(x) = \lim_{n \to \infty} f_n(x)$ exists for each $x \in X$. Then f is a continuous linear transformation of X into Y.*

Proof. Clearly, f is a homomorphism and the sequence $\{f_n(x)\}$ is bounded. By Theorem 1, the sequence $\{f_n\}$ is equicontinuous. Let V be any neighbourhood of 0 in Y. Then there exist a closed neighbourhood $V_0 \subseteq V$ and a neighbourhood U of 0 in X such that $f_n(U) \subseteq V_0$ $(n = 1, 2, \ldots)$. Now, for any $x \in U$,

$$f(x) = \lim_{n\to\infty} f_n(x) \in \overline{V}_0 = V_0$$

and so $f(U) \subseteq V$; that is, f is continuous. ∎

For X, as in Theorem 1, let $M \subset X^*$ be ω^*-bounded (i.e., $\sup\{|f(x)| : f \in M\} < \infty$ for every $x \in X$). Then M is pointwise bounded in X^* and so bounded by Theorem 1.

In the same way, some other results purely dependent on the classical uniform boundedness principle can be adopted from [11], [26], and [30] in this general setting.

As another application of Theorem A, we indicate how the Banach-Steinhaus theorem on condensation of singularities ([27], Corollary 3, p. 121) may be derived from it.

Theorem 3 *Let $\{U_{n,m} : n, m = 1, 2, \ldots\}$ be a double sequence of bounded linear transformations of a Banach space X into a Banach space Y such that for each $m = 1, 2, \ldots, \limsup_n \|U_{n,m}\| = +\infty$. Then there is a set S of the second category in X such that, for each x in S and each $m = 1, 2, \ldots$, $\limsup_n \|U_{n,m}(x)\| = +\infty$.*

Proof. For each positive integer m, n define $p_{n,m}(x) = \|U_{n,m}(x)\| \quad (x \in X)$. It is easy to see that each $p_{n,m}$ is a continuous sub-additive functional on X. For each positive integer m, define

$$Z_m = \{z \in X : \limsup_n p_{n,m}(z) = +\infty\}$$

and

$$Z = \bigcap_{m=1}^{\infty} Z_m.$$

The condition $\limsup_n \|U_{n,m}\| = +\infty$ implies that, for each $m = 1, 2, \ldots$, $\limsup_n (\sup_{x\in U} p_{n,m}(x)) = +\infty$ for each neighbourhood U of 0 in X, and therefore by Theorem A (with $x_n = 0$ for all positive integers n), Z_m is a residual G_δ-set. It follows that Z is a residual G_δ-set. Since X is a Banach space, therefore $Z = \{z \in X : \limsup_n \|U_{n,m}(z)\| = +\infty$ for $m = 1, 2, \ldots\}$ is of second category and is the desired set S. ∎

A locally convex space in which a norm is available, is said to have the property (N). For example, a normed space and the space $(X^*, \omega^*)^*$ where X is a locally convex space have the property (N).

In the remainder of this section it is assumed that X is a complete locally convex space with the property (N).

We need the following pair of lemmas:

Lemma 4 *The following statements are equivalent for a subset A of X:*

(a) A is norming for X^*

(b) $\overline{co}(\pm A)$ *is norming for* X^*

(c) there exits a $\delta > 0$ *such that* $\overline{co}(\pm A) \supset \delta B_X$.

Proof. The only non-trivial implication is (a) $\Rightarrow$ (c).

Assume that $\overline{co}(\pm A) \subset \delta B_X$ for all $\delta > 0$. Consider a sequence $\{x_n\}$ in $X \setminus \overline{co}(A)$ converging to 0. For each $n, x_n \notin \overline{co}(\pm A)$, an absolutely convex subset of X, so by Proposition 2.1 (see also Theorem 4.25 in [11]) there exists $x_n^* \in X^*$ such that

$$|x_n^*(x_n)| > \sup_{a \in \overline{co}(\pm A)} |x_n^*(a)| \geq \sup_{a \in A} |x_n^*(a)|.$$

Now using (a), we may obtain a $\delta > 0$ satisfying

$$|x_n^*(x)| > \inf_{x_n^* \in S_{X^*}} \sup_{a \in A} |x_n^*(a)| > \delta.$$

Plainly the choice of $\{x_n\}$ implies that $|x_n^*(x_n)| < \delta$ for all $\delta > 0$ and $n \geq n_0$. This contradiction proves the result ■

The following analogous result for the dual space X^* is easy to verify.

Lemma 5 *The following statements are equivalent for a subset* B *of* X^**:*

(a) B *is norming for* X

(b) $\overline{co}(\pm B)$ *is norming for* X

(c) there exists a $\delta > 0$ *such that* $\overline{co}^{\omega^*}(\pm B) \supseteq \delta B_{X^*}$.

Lemma 6 *If* A *is a subset of the second category in* X*, then* A *is thick.*

Proof. Let $\{A_i\}$ be an increasing sequence with $A = \bigcup_{i=1}^{\infty} A_i$. As A is of second category, some $\overline{A_m}$ contains a ball $S_r(x)$. Hence, it follows that $S_r(0) \subseteq \overline{co}(\pm A_m)$. This implies, by Lemma 4 (with $\delta = 1$), A_m is norming. Since $\{A_i\}$ is arbitrary, therefore A must be thick. ■

The classical uniform boundedness principle holds beyond sets of the second category; this is the case with the set S of characteristic functions in the unit sphere of the function space $B(\mathcal{A})$ where $\mathcal{A}$ is a σ-algebra of sets (cf. [6]). Note that S is merely nowhere dense. We continue this theme and generalize Theorems 1 and 2 and Proposition 2.2 of Nygaard [23] in the sense that the domain of transformations is a thick set in X and its dual space X^*. Our methods are based on those used by Nygaard [22-23].

Theorem 7 *Let* A *be a thick subset of* X*. Suppose that* Y *is a Hausdorff locally convex space and* $\{f_n\}$ *a sequence of continuous linear transformations of* X *into* Y *such that* $\{f_n(x)\}$ *is bounded for each* $x \in A$*. Then the sequence* $\{f_n\}$ *is equicontinuous.*

Proof. Suppose that $\{f_n\}$ is pointwise bounded on A, that is, $\sup_n p(f_n(x)) < \infty$ for all $x \in A$ and each $p \in P$. Put $A_m = \{x \in A : \sup_n p(f_n(x)) \leq m$ for each $p \in P\}$. The sequence $\{A_m\}$ of sets is increasing with $A = \bigcup_{i=1}^{\infty} A_i$. As A is thick, some A_k is norming. Thus, by Lemma 4, there exists a $\delta > 0$ such that $\delta B_X \subseteq \overline{co}(\pm A_k)$. This together with the definition of A_m implies that $\delta p(f_n) = \sup_{x \in \delta S_X} p(f_n(x)) \leq \sup_{x \in \overline{co}(\pm A_k)} p(f_n(x)) \leq k$. Hence, $\sup_n p(f_n) \leq \frac{k}{\delta} < \infty$ as desired. ■

Remark 8 *Theorem 7 extends Proposition 2.2 of Nygaard [23].*

Theorem 9 *Let B be a thick subset of X^*. Suppose that Y is a Hausdorff locally convex space and $\{f_n^*\}$ a sequence of continuous linear transformations of X^* into Y^* such that $\{f_n^*(x^*)\}$ is bounded for each x^* in B. Then the sequence $\{f_n^*\}$ is equicontinuous.*

Proof. Follows pattern of the proof of Theorem 7; the only difference is that we consider

$$A_m = \{x^* \in B : \sup_n q^*(f_n^*(x^*)) \leq m \text{ for each } q^* \in Q^*\}$$

and use Lemma 5 and the ω^*-continuity of f_n^*. ■

The proofs of the following corollaries follow pattern of the proof of Theorem 2 and so will be omitted.

Corollary 10 *Let X, A and Y be as in Theorem 7 and $\{f_n\}$ be a sequence of continuous linear transformations of X into Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Then f is a continuous linear transformation of X into Y.*

Corollary 11 *Let X^*, B and Y^* be as in Theorem 9 and $\{f_n^*\}$ be a sequence of continuous linear transformations of X^* into Y^* such that $f^*(x^*) = \lim_{n\to\infty} f_n^*(x^*)$ exists for each $x^* \in X^*$. Then f^* is a continuous linear transformation of X^* into Y^*.*

We now establish the Nikodym boundedness theorem in more general settings in relation to the domain, range and nature of mappings.

Theorem 1 due to Drewnoski [7] is proved in the context of a quasi-normed group; we observe that his proof can be readily modified to the case of any commutative Hausdorff topological group G to obtain a principle of equicontinuity type result for group measures as follows:

Theorem 12 *Let M be a family of exhaustive G-valued measures on a σ-ring $\mathcal{R}$ such that for each $E \in \mathcal{R}, \{\mu(E) : \mu \in M\}$ is a bounded subset of G. Then $\{\mu(E) : E \in \mathcal{R}, \mu \in M\}$ is a bounded subset of G.*

The assumption that $\mathcal{R}$ is a σ-ring is essential in the above theorem (see [7], Example, p. 117).

Valuable contributions have been made in special but very important field of submeasures with values in a commutative ℓ-group (see [2] and [17] and the references therein). In the next result, we prove Theorem 12 for group-valued submeasures to get the following principle of equicontinuity which generalizes Theorem 1 of Drewnowski [7].

Theorem 13 *Let (G, q) be an ℓ-quasi-normed group and M be a family of G-valued submeasures on a σ-ring $\mathcal{R}$ such that*

$$\sup_{\mu \in M} q(\mu(E)) < +\infty$$

for each E in $\mathcal{R}$. Then $\sup_{\substack{\mu \in M \\ E \in \mathcal{R}}} q(\mu(E)) < +\infty$.

Proof. Let H be the group of all G-valued mappings on M. Clearly, H is a commutative partially ordered group, the order being $f \leq g$ if and only if $f(\mu) \leq g(\mu)$ for all $\mu \in M$. Define the functional ϕ on H by

$$\phi(f) = \sup_{\mu \in M} q(f(\mu)).$$

Note that ϕ is an extended real-valued quasi-norm on H with $\phi(f) \leq \phi(g)$ for $0 \leq f \leq g$. Define a mapping $\nu : \mathcal{R} \to H$ by

$$\nu(E)(\mu) = \mu(E).$$

Clearly, ν is an H-valued submeasure on $\mathcal{R}$.

Suppose not; then with the above notation, $\sup_{E \in \mathcal{R}} \phi(\nu(E)) = +\infty$. Thus, for each positive integer n, there exists a set E_n in $\mathcal{R}$ such that $\phi(\nu(E_n)) > n$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Now $E \in \mathcal{R}$ and $\phi(\nu(E)) = +\infty$. This implies that $\sup_{\mu} q(\mu(E)) = +\infty$, which contradicts the hypothesis. Hence, $\sup_{\substack{\mu \in M \\ E \in \mathcal{R}}} q(\mu(E))$ is finite. ■

Finally, every σ-algebra of sets on a finite set S is a topology but not conversely. Thus, the result to follow extends the domain of maps in ([6], Corollary 2, p. 16) and Proposition 2.1 [23], simultaneously.

Theorem 14 *Let A be a thick subset of X. If $\{f_n\}$ is a sequence of continuous linear functionals on X such that $\{f_n(x)\}$ is bounded for each x in A, then the sequence $\{f_n\}$ is equicontinuous.*

Proof. Take Y as the space of scalars in the proof of Theorem 7. ■

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References

[1] T.A. Abrahamsen, O. Nygaard and M. Põldvere, On weak integrability and boundedness in Banach spaces, J. Math. Anal. Appl. 314(2006), 67–74.

[2] A.Avallone-A. Valente, A decomposition theorem for submeasures, Atti. Sem. Mat. Fis. Univ. Modena, XLIII (1995), 81–90.

[3] N. Bourbaki, Topological Vector Spaces, Chapters 1-5, Springer-Verlag, (2003).

[4] J. Daneš, On Banach-Steinhaus theorem, Bolletino U.M.I. (5), 13-A (1976), 282–286.

[5] R.B. Darst, On a theorem of Nikodym with applications to weak convergence and von Neumann algebras, Pacific J. Math., 23(1967), 473–477.

[6] J. Diestel and J.J. Uhl, Jr., Vector Measures, American Mathematical Society, Providence, Mathematical Surveys, 15(1977).

[7] L. Drewnoski, Uniform boundedness principle for finitely additive vector measures, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 21 (1973), 115–118.

[8] L. Drewnowski, Equivalnce of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems, ibid. 20(1972), 725–731.

[9] N. Dunford and J.T. Schwartz, Linear Operators part I, John Wiley and Sons, (1958).

[10] Y. Eidelman, V. Milman and A. Tsolomitis, Functional Analysis, An Introduction, American Mathematical Society, Providence, (2004).

[11] M. Fabian et al., Functional Analysis and Infinite Dimensional Geometry, Springer-Verlag, (2001).

[12] V.P. Fonf, On exposed and smooth points of convex bodies in Banach spaces. Bull. London Math. Soc., 28 (1996), 51–58.

[13] V.P. Fonf, Weakly extremal properties of Banach spaces, Math. Notes, 45 (1989), 488-494.

[14] D.H. Fremlin, Topological Riesz Spaces and Measure Theory, Cambridge University Press, (1974).

[15] M.I. Kadets and V.P. Fonf, Two theorems on the massiveness of the boundary in a reflexive Banach space, Funktsional Anal. i Prilozhen. 17(1983), 77–78; English translation in: Funct. Anal. Appl. 17(1983), 227–228.

[16] A.R. Khan and K. Rowlands, On a theorem of Daneš and the principle of equicontinuity, Bollettino U.M.I. (6), 5-A (1986), 211–215.

[17] A.R. Khan and K. Rowlands, A decomposition theorem for submeasures, Glasgow Math. J., 26 (1985), 67–74.

[18] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, (1978).

[19] I. Labuda, Sur quelques generalisations des theorems de Nikodym et de Vitali-Hahn-Saks, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. 20(1972), 447–456.

[20] A.G. Mikusinski, On a theorem of Nikodym on bounded measures, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. 19(1971), 441–444.

[21] O.M. Nikodym, Sur les familles bornées de fonctions parfaitement additives d'ensemble abstrait, Monatsh. Math. Phys. 40(1933), 418–426.

[22] O. Nygaard, Boundedness and surjectivity in normed spaces, Int. J. Math. & Math. Sc., 32 (2002), 149–165.

[23] O. Nygaard, A strong uniform boundedness principle in Banach spaces, Proc. Amer. Math. Soc., 129 (2001), 861–863.

[24] M. Pasteka, A note about the submeasures and Fermat last theorem, Ricerche di Matematica, fasc. i (1994), 79–90.

[25] W. Rudin, Functional Analysis, Second Edition, McGraw-Hill, (1991).

[26] M. Schechter, Principles of Functional Analysis, Second Edition, American Mathematical Scoiety, Providence, (2002).

[27] C. Swartz, An Introduction to Functional Analysis, Marcel Dekker, Inc. (1992).

[28] G.E.F. Thomas, Le theoreme de Lebesgue-Nikodym pour les measures vectorielles et applications 1-some-mantes, C.R. Acad. Sci. Paris, (1970), 872–875.

[29] L. Waelbroeck, Topological Vector Spaces and Algebras, Lecture Notes in Mathematics, No. 230, Springer-Verlag, (1971).

[30] K. Yosida, Functional Analysis, Fourth Edition, Springer-Verlag, (1974).

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