A DECOMPOSITION THEOREM FOR SUBMEASURES

by A. R. KHAN and K. ROWLANDS

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1. Introduction. In recent years versions of the Lebesgue and the Hewitt–Yosida decomposition theorems have been proved for group-valued measures. For example, Traynor [**4**], [**6**] has established Lebesgue decomposition theorems for exhaustive group-valued measures on a ring using (1) algebraic and (2) topological notions of continuity and singularity, and generalizations of the Hewitt–Yosida theorem have been given by Drewnowski [**2**], Traynor [**5**] and Khurana [**3**]. In this paper we consider group-valued submeasures and in particular we have established a decomposition theorem from which analogues of the Lebesgue and Hewitt–Yosida decomposition theorems for submeasures may be derived. Our methods are based on those used by Drewnowski in [**2**] and the main theorem established generalizes Theorem 4.1 of [**2**].

2. Notation and terminology. Let G be a commutative lattice group (abbreviated to l-group). A quasi-norm (resp. norm) q on G is said to be an *l-quasi-norm* (*l-norm*) if $q(x)\leq q(y)$ for all x, y in G with $|x|\leq|y|$. A G-valued function μ defined on a ring $\mathcal{R}$ of subsets of a set X is said to be a submeasure if $\mu(\varnothing)=0$, $\mu(E\cup F)\leq\mu(E)+\mu(F)$ for all E, F in $\mathcal{R}$ with $E\cap F=\varnothing$, and $\mu(E)\leq(F)$ for all E, F in $\mathcal{R}$ with $E\subseteq F$. A G-valued submeasure μ on $\mathcal{R}$ is said to be *exhaustive* if and only if, for any disjoint sequence $\{E_n\}$ in $\mathcal{R}$, $\lim_{n\to\infty}\mu(E_n)=0$ in (G,q). An l-group G is said to be *order complete* if every bounded increasing net in G has a supremum. An l-quasi-norm q on G is said to be *order continuous* if $\varnothing\subset A\uparrow x$ in $G^+=\{x\in G: x\geq 0\}$ implies $q(x)=\sup\{q(y): y\in A\}$ and $B\downarrow x$ in G^+ implies $q(x)=\inf\{q(y): y\in B\}$.

Let $\mathcal{D}$ denote a collection of pairwise disjoint sets in $\mathcal{R}$ and let Δ be the set of all such collections. If $\mathcal{D}_1$, $\mathcal{D}_2\in\Delta$, then we write $\mathcal{D}_1\leq\mathcal{D}_2$ if and only if $\mathcal{D}_2$ is a refinement of $\mathcal{D}_1$. With each $E\in\mathcal{R}$ we associate members of $\mathcal{D}$; the collection of all such pairs $(E,\mathcal{D})$ is denoted by $\mathcal{G}$ and we let

$$\mathcal{G}(E)=\{\mathcal{D}\in\Delta:(E,\mathcal{D})\in\mathcal{G}\}\quad\text{and}\quad\Delta_{\mathcal{G}}=\bigcup_{E\in\mathcal{R}}\mathcal{G}(E)$$

In the sequel we use $\bigcup\mathcal{D}$ to mean the set theoretic union of the members of $\mathcal{D}$. Following Drewnowski's terminology ([**2**], Definition 2.1), the collection $\mathcal{G}$ is said to be an *additivity* on $\mathcal{R}$ if it satisfies the following conditions:

(a) $\Delta_f\subseteq\Delta_{\mathcal{G}}$, where Δ_f consists of those collections $\mathcal{D}$ which have only a finite number of members;

(b) if $E\in\mathcal{R}$ and $\mathcal{D}\in\mathcal{G}(E)$, then $\bigcup\mathcal{D}=E$;

(c) if $E\in\mathcal{R}$, $\mathcal{D}_1$, $\mathcal{D}_2\in\mathcal{G}(E)$, then $\mathcal{D}_1\cap\mathcal{D}_2\in\mathcal{G}(E)$, where $\mathcal{D}_1\cap\mathcal{D}_2=\{D_1\cap D_2: D_i\in\mathcal{D}_i, i=1,2\}$.

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