COMMON FIXED POINTS FROM BEST SIMULTANEOUS APPROXIMATIONS

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Abstract

We obtain some results on common fixed points from the set of best simultaneous approximations for a map T which is asymptotically (f,g)-nonexpansive where (T,f) and (T,g) are not necessarily commuting pairs. Our results extend and generalize recent results of Chen and Li [1], Jungck and Sessa [8], Sahab et al. [13], Sahney and Singh [14], Singh [15, 16] and Vijayaraju [17] and many others.

1. Introduction and Preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|.\|)$. The set $P_M(u) = \{x \in M : \|x-u\| = dist(u,M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u,M) = inf\{\|y-u\| : y \in M\}$. Suppose that A and G are bounded subsets of X. Then we write

$$r_G(A) = inf_{g \in G} sup_{a \in A} \parallel a - g \parallel$$

$$cent_G(A) = \{g_0 \in G : sup_{a \in A} \parallel a - g_0 \parallel = r_G(A)\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t. G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t. G. If $A = \{u\}$, then $r_G(A) = dist(u,G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G. We also refer the reader to Milman [12] and Vijayaraju [17] for further details. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(M)$ to denote the closure of a set M and $wcl(M)$ to denote the weak closure of a set M. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an *(f,g)-contraction* if there exists $0 \leq k < 1$ such that $\|Tx - Ty\| \leq k\|fx - gy\|$ for any $x, y \in M$. If $k = 1$, then T is called (f,g)-nonexpansive. The map T is called *asymptotically (f,g)-nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|fx - gy\|$ for all $x, y \in M$ and $n = 1, 2, 3, ...$; if $g = f$, then T is called *asymptotically f-nonexpansive* [17]. The map T is called *uniformly asymptotically regular* [17] on M, if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1} x\| < \eta$ for all $n \geq N$ and all $x \in M$. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ $(x = fx = Tx)$. The set of coincidence points of f and T is denoted by $C(f,T)$. The pair $\{f, T\}$ is called: (1) *commuting* if $Tfx = fTx$ for all $x \in M$, (2) *compatible* (see [7]) if $\lim_n \|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M; (3) *weakly compatible* if they commute at their coincidence points, i.e.,if $fTx = Tfx$ whenever $fx = Tx$. The set M is called *q-starshaped* with $q \in M$, if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. The map f defined on a q-starshaped set M is called *affine* if

$$f((1-k)q + kx) = (1-k)fq + kfx, \quad for \quad all \quad x \in M.$$

[1]Key Words: Banach operator pair; Asymptotically (f,g)-nonexpansive maps; Best simultaneous approximation.
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