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Department of Mathematics and Statistics
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia
Emails: anwarj@kfupm.edu.sa, abujiya@kfupm.edu.sa

Abstract: Some centered moments of the bivariate chi-square distribution are derived by the use of raw product moments. Standardized moments up to the third order are calculated for the distribution. In case the components of bivariate chi-square distribution are uncorrelated, the moments, as expected, are in agreement with the resulting situation of independence. The results are also in agreement with the case when the degrees of freedom converges to infinity.

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1. Introduction

Fisher (1915) derived the distribution of mean-centered sum of squares and sum of products in order to study the distribution of correlation coefficient from a bivariate normal sample. Let S_1^2 and S_2^2 be the usual sample variances of a sample of size N from a usual bivariate normal distribution given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \xi \\ \theta \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}\right).$$

Then the joint distribution $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ where $m = N-1 > 2$ is called the bivariate chi-square distribution after Krishnaiah, Hagis and Steinberg (1963). For $\sigma_1 = \sigma_2$, Finney (1938) derived the sampling distribution of the square root of the ratio of correlated chi-squares variables ($T = \sqrt{U/V}$) directly from the joint distribution of sample variances and correlation coefficient. He compared the variability of the measurements of standing height and stem length for different age groups of schoolboys by his method and by Hirschfeld (1937).

Joarder (2008) derived the following density function of the bivariate chi-square distribution.

Theorem 1.1: The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its density function is given by

$$f_1(u,v)=\frac{(uv)^{m/2-1}\exp\left(-\frac{u+v}{2-2\rho^2}\right)}{2^m\sqrt{\pi}\,\Gamma(m/2)(1-\rho^2)^{m/2}}\sum_{k=0}^{\infty}\left[1+(-1)^k\right]\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left((k+1)/2\right)}{k!\Gamma\left((k+m)/2\right)}, \tag{1.1}$$

$m=N-1>2$, $-1<\rho<1$.

In case $\rho=0$, the above density function would be that of the product of two independent chi-square random variables given by

$$f_1(u,v)=\frac{(uv)^{m/2-1}e^{-(u+v)/2}}{2^m\ \Gamma^2(m/2)},\ u>0, v>0. \tag{1.2}$$

There are also a number of other bivariate chi-square and gamma distributions excellently reviewed by Kotz, Balakrishnan and Johnson (2000).

The product moments (also called raw product moments or product moments around zero) of order a and b for two random variables X and Y are defined by $\mu'(a,b)=E(X^aY^b)$ while the centered product moments (sometimes called corrected product moments or central mixed moments) are defined by

$$\mu(a,b)=E\left[(X-\xi)^a(Y-\theta)^b\right] \tag{1.3}$$

where $\xi=E(X),\theta=E(Y)$. See for example, Johnson, Kotz and Kemp (1993, 46). Evidently $\mu'(a,0)=E\left(X^a\right)$ is the a-th moment of X, and $\mu'(0,b)=E\left(Y^b\right)$ is the b-th moment of Y. In case X and Y are independent $\mu'(a,b)=E(X^a)E(Y^b)=\mu'(a,0)\mu'(0,b)$. The correlation coefficient ρ $(-1<\rho<1)$ between X and Y is given by

$$\rho_{XY}=\frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}. \tag{1.4}$$

Obviously $\mu(2,0)=E(X-\xi)^2$ and $\mu(0,2)=E(Y-\theta)^2$ are the variances of X and Y respectively.

In a series of papers, Mardia (1970; 1974; 1975) defined and discussed the properties of measures for kurtosis and skewness based on Mahalanobis distance. In this paper, we have derived raw product moments in Corollary 2.1 and centered product moments in Theorem 2.2. Centered product moments of the bivariate chi-square distribution deemed essential have been expressed in terms of raw product moments in Section 2. Standadrized moments for any bivariate distribution is presented in Section 3. As it is difficult to derive distribution of Mahalanobis distance for bivariate chi-square distribution, we derive standardized moments (equivalently Mahalanobis moments) in terms of centered product moments in Section 4. The main contribution in the paper is the explicit expression of the second and third order standardized moments of the bivariate chi-square distribution given in Theorem 4.1.

It is worth mentioning that the second standardized moment accounts for kurtosis. In case the components of bivariate chi-square distribution are uncorrelated ($\rho = 0$), the moments, as expected, are in agreement with the resulting situation of independence. The results are also in agreement with the case when the degrees of freedom converges to infinity.

2. Product Moments of the Bivariate Chi-square Distribution

The following theorem is due to Joarder (2008).

Theorem 2.1: Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1. Then the (a,b)-th product moment of U and V is given by

$$E(U^a V^b) = \frac{2^{a+b}(1-\rho^2)^{a+b}}{L(m,\rho)} \sum_{k=0}^{\infty} [1+(-1)^k] \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2}+a\right) \Gamma\left(\frac{k+m}{2}+b\right) \frac{\Gamma\left((k+1)/2\right)}{\Gamma\left((k+m)/2\right)}$$

where $L(m,\rho) = \sqrt{\pi}\,\Gamma(m/2)(1-\rho^2)^{-m/2}$, $m > 2$, and $-1 < \rho < 1$.

It is obvious that $E(U^a V^b) = E(U^b V^a)$ with $E(U) = E(V) = m$. If $\rho = 0$, then $E(U^a V^b) = E(U^a)E(V^b)$, where U is distributed as chi-square with m degrees of freedom, and so is V. Note that $E(U^a) = 2^a\ \Gamma((m/2)+a)/\Gamma(m/2), a = 1, 2, \cdots$. Some raw product moments of the bivariate chi-square distribution that are needed for the standardized moments up to the third order are provided in the following corollary.

Corollary 2.1: Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1. Then we have the following specific raw product moments of U and V :

$$\begin{aligned}
E(UV) &= m(m+2\rho^2), \\
E(UV^2) &= m(m+2)(m+4\rho^2), \\
E(UV^3) &= m(m+2)(m+4)(m+6\rho^2),
\end{aligned}$$

$$E(UV^4) = m(m+2)(m+4)(m+6)(m+8\rho^2),$$
$$E(UV^5) = m(m+2)(m+4)(m+6)(m+8)(m+10\rho^2),$$
$$E(U^2V^2) = m(m+2)\left[8\rho^4+8(m+2)\rho^2+m(m+2)\right],$$
$$E(U^2V^3) = m(m+2)(m+4)\left[24\rho^4+12(m+2)\rho^2+m(m+2)\right],$$
$$E(U^2V^4) = m(m+2)(m+4)(m+6)\left[48\rho^4+16(m+2)\rho^2+m(m+2)\right],$$
$$E(U^3V^3) = m(m+2)(m+4)\left[48\rho^6+72(m+4)\rho^4+18(m+2)(m+4)\rho^2+m(m+2)(m+4)\right].$$

Proof. Put specific values of a and b in Theorem 2.1 above and simplify by the generalized hypergeometric function.

If $\rho = 0$, the above moments coincide with those of the probability model in (1.2). Since $E(U) = E(V) = m$, the centered product moments of order a and b of the chi-square random variables U and V is given by

$$\mu(a,b) = E\left[(U-m)^a(V-m)^b\right]. \tag{2.1}$$

By expanding (2.1) for specific values of a and b, we have centered product moments of the bivariate chi-square distribution expressed below in terms of raw product moments.

$$\mu(1,1) = E(UV) - m^2,$$
$$\mu(2,0) = E(U^2) - m^2,$$
$$\mu(2,1) = -2mE(UV) + E(U^2V) - mE(U^2) + 2m^3,$$
$$\mu(2,2) = -3m^4 + 2m^2E(U^2) + 4m^2E(UV) - 4mE(U^2V) + E(U^2V^2),$$
$$\mu(3,0) = E(U^3) - 3mE(U^2) + 2m^3,$$
$$\mu(3,1) = 3m^2E(UV) - 3mE(U^2V) + E(U^3V) + 3m^2E(U^2) - mE(U^3) - 3m^4,$$
$$\mu(3,2) = E(U^3V^2) - 3mE(U^2V^2) - 2mE(U^3V) + 9m^2E(U^2V) - 4m^3E(U^2)$$
$$- 6m^3E(UV) + m^2E(U^3) + m^5 - 3m^4,$$
$$\mu(3,3) = -5m^6 + 9m^4E(UV) - 18m^3E(U^2V) + 6m^4E(U^2) + 9m^2E(U^2V^2) - 2m^3E(U^3)$$
$$+ 6m^2E(U^3V) - 6mE(U^3V^2) + E(U^3V^3),$$
$$\mu(4,0) = -3m^4 + 6m^2E(U^2) - 4mE(U^3) + E(U^4),$$
$$\mu(4,1) = -4m^3E(UV) + 6m^2E(U^2V) - 4mE(U^3V) + E(U^4V) - 6m^3E(U^2)$$
$$+ 4m^2E(U^3) - mE(U^4) + 4m^5,$$
$$\mu(4,2) = -5m^6 + 7m^4E(U^2) - 4m^3E(U^3) + m^2E(U^4) + 8m^4E(UV) - 16m^3E(UV^2)$$
$$+ 8m^2E(U^3V) - 2mE(U^4V) + 6m^2E(U^2V^2) - 4mE(U^3V^2) + E(U^4V^2),$$

$$\mu(5,0) \;=\; 4m^5 - 10m^3E(U^2) + 10m^2E(U^3) - 5mE(U^4),$$

$$\mu(5,1) \;=\; 5m^4E(UV) - 10m^3E(U^2V) + 10m^2E(U^3V) - 5mE(U^4V) + E(U^5V) - 5m^6$$
$$+10m^4E(U^2) - 10m^3E(U^3) + 5m^2E(U^4) - mE(U^5),$$

$$\mu(6,0) \;=\; -5m^6 + 15m^4E(U^2) - 20m^3E(U^3) + 15m^2E(U^4) - 6mE(U^5) + E(U^6).$$

Since $U \sim \chi^2_m$, we have

$$\mu(2,0) = 2m, \quad \mu(4,0) = 12m^2 + 48m, \quad \mu(6,0) = 120m^3 + 2080m^2 + 3840m\,. \tag{2.2}$$

Theorem 2.2: Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1. Then for $m > 2$ and $-1 < \rho < 1$, we have the following specific centered product moments of U and V:

(*i*) $\mu(1,1) \;=\; 2m\rho^2,$

(*ii*) $\mu(2,2) \;=\; (8\rho^4 + 4)m^2 + (16\rho^4 + 32\rho^2)m,$

(*iii*) $\mu(3,1) \;=\; 12\rho^2m^2 + 48\rho^2m,$

(*iv*) $\mu(3,3) \;=\; (48\rho^6 + 72\rho^2)m^3 + (288\rho^6 + 1152\rho^4 + 576\rho^2 + 64)m^2$
$\qquad + (384\rho^6 + 2304\rho^4 + 1152\rho^2)m,$

(*v*) $\mu(4,2) \;=\; (96\rho^4 + 24)m^3 + (1344\rho^4 + 640\rho^2 + 96)m^2 + (2304\rho^4 + 1536\rho^2)m.$

(*vi*) $\mu(5,1) \;=\; 120\rho^2m^3 + 2080\rho^2m^2 + 3840\rho^2m$

Proof. The centered product moments in the theorem follow from the expression in (2.1) by using the raw product moments in Corollary 2.1. Let us prove *(v)*. Plugging in $E(U^2)$, $E(U^3)$, $E(U^4)$, $E(UV)$, $E(UV^2)$, $E(U^3V)$, $E(U^4V)$, $E(U^2V^2)$, $E(U^3V^2)$ and $E(U^4V^2)$ in the expression

$$\mu(4,2) \;=\; -5m^6 + 7m^4E(U^2) - 4m^3E(U^3) + m^2E(U^4) + 8m^4E(UV) - 16m^3E(UV^2)$$
$$+8m^2E(U^3V) - 2mE(U^4V) + 6m^2E(U^2V^2) - 4mE(U^3V^2) + E(U^4V^2),$$

we get

$$\mu(4,2) = -5m^6 + 7m^4[m(m+2)] - 4m^3[m(m+2)(m+4)] + m^2[m(m+2)(m+4)(m+6)] + 8m^4[m(m+2\rho^2)]$$
$$-16m^3[m(m+2)(m+4\rho^2)] + 8m^2[m(m+2)(m+4)(m+6\rho^2)] - 2m[m(m+2)(m+4)(m+6)(m+8\rho^2)]$$
$$+6m^2\times m(m+2)[8\rho^4 + 8(m+2)\rho^2 + m(m+2)] - 4m\times m(m+2)(m+4)[24\rho^4 + 12(m+2)\rho^2 + m(m+2)]$$
$$+m(m+2)(m+4)(m+6)[48\rho^4 + 16(m+2)\rho^2 + m(m+2)]$$

which simplifies to (*v*). If $\rho = 0$, the above corrected moments coincide with those of the probability model in (1.2).

3. Mahalanobis Moments for Bivariate Distributions

Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ have a bivarite distribution with mean vector $\begin{pmatrix} \xi \\ \theta \end{pmatrix}$ and covariance matrix $\begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix}$ where $\mu(a,b) = E[(X-\xi)^a(Y-\theta)^b]$. The Mahalanobis moments or standardized moments of the bivariate distribution are given by $\beta_i = E(Q^i), (i = 1, 2, \cdots)$ where

$$Q = W'\Omega^{-1}W\ ,\ W' = (X-\xi,\ Y-\theta)\,. \tag{3.1}$$

Note that

$$\beta_1 = E(trW'\Omega^{-1}W) = E(tr\Omega^{-1}WW') = tr[\Omega^{-1}E(WW')] = tr(\Omega^{-1}\Omega) = tr(I_2) = 2. \tag{3.2}$$

Obviously, the first standardized moment for any distribution is the dimension of the random variable in question. The higher order Mahalanobis moments or standardized moments of the bivariate distribution can be calculated if the specific form of the distribution is known. The second standardized moment β_2 is the coefficient of kurtosis in the sense of Mardia (1970; 1974; 1975). The following result is due to Joarder (2007).

Theorem 3.1: Let X and Y have a bivariate distribution with $E(X^aY^b) = E(X^bY^a)$, the correlation between them be ρ and Q be defined by (3.1). Then we have the following:

(i) $\quad \mu^2(0,2)(1-\rho^2)^2E(Q^2) \;=\; 2\mu(4,0)+(4\rho^2+2)\mu(2,2)-8\rho\mu(3,1),$

(ii) $\quad \mu^3(0,2)(1-\rho^2)^3E(Q^3) \;=\; 2\mu(6,0)-(8\rho^3+12\rho)\mu(3,3)-12\rho\mu(5,1)+(24\rho^2+6)\mu(4,2).$

Corollary 3.1: Let X and Y have a bivariate distribution and Q be defined by (3.1). If X and Y are independently and identically distributed, then we have the following:

(i) $\quad E(Q^2) \;=\; 2\left(1+\dfrac{\mu(4,0)}{\mu^2(2,0)}\right),$

(ii) $\quad E(Q^3) \;=\; 2\left(\dfrac{\mu(6,0)}{\mu^3(2,0)}+3\dfrac{\mu(4,0)}{\mu^2(2,0)}\right).$

4. The Main Result

Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1. Since $\mu(2,0)=2m,\ \mu(1,1)=2m\rho^2,\ \mu(0,2)=2m,$ the covariance matrix between U and V is given by

$$\Omega=\begin{pmatrix} 2m & 2m\rho^2 \\ 2m\rho^2 & 2m \end{pmatrix}$$

while the correlation coefficient (see 1.4) between U and V is given by

$$\rho_{U,V}=\frac{E[(U-E(U))(V-E(V))]}{\sqrt{E(U-E(U))^2E(V-E(V))^2}}=\frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}=\rho^2.$$

The Mahalanobis moments or standardized moments of the bivariate chi-square distribution are given by $\beta_i=E(Q^i),(i=1,2,\cdots)$ where

$$Q=W'\Omega^{-1}W\ ,W'=(U-m\ \ V-m)\,. \tag{4.1}$$

By (3.2), the first order Mahalanobis moments or standardized moments of the bivariate chi-square distribution is given by 2.

Theorem 4.1: Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1, and Q be defined by (4.1). Then for $m>2$ and $-1<\rho<1$, the second and the third order Mahalanobis moments of the bivariate chi-square distribution are given by

$$(i)\qquad m(1-\rho^4)^2E(Q^2)\ =\ 8m(1-\rho^4)^2+8(3+2\rho^2-11\rho^4+4\rho^6+2\rho^8),\ \text{ and}$$

$$\begin{aligned}(ii)\qquad m^2(1+\rho^2)E(Q^3)\ =\ & 48(1+\rho^2)^3m^2+16(18\rho^6+126\rho^4+135\rho^2+37)m\\ & +192(2\rho^6+18\rho^4+16\rho^2+5)\end{aligned}$$

respectively.

Proof. Since the correlation between U and V is $\rho_{U,V}=\rho^2$, we replace $\rho_{X,Y}=\rho$ by $\rho_{U,V}=\rho^2$ in the expressions in Theorem 3.1 (i). Moreover since $E(U^aV^b)=E(U^bV^a)$, it follows from Theorem 3.1 (i) that

$$\mu^2(0,2)(1-\rho^4)^2E(Q^2)\ =\ 2\mu(4,0)+(4\rho^4+2)\mu(2,2)-8\rho^2\mu(3,1).$$

By plugging in the moments from Section 2, we have the following:

$$4m^2(1-\rho^4)^2E(Q^2)$$
$$= 2(12m^2+48m)+(4\rho^4+2)(8\rho^4m^2+16\rho^4m+32\rho^2m+4m^2)-8\rho^2(12m^2\rho^2+48m\rho^2)$$
$$= 4(8\rho^8-16\rho^4+8)m^2+4(16\rho^8+32\rho^6-88\rho^4+16\rho^2+24)m$$

which is equivalent to (*i*).

Again, replacing $\rho_{X,Y}=\rho$ by $\rho_{U,V}=\rho^2$ in the expressions in Theorem 3.1 (ii), we have

$$\mu^3(0,2)(1-\rho^4)^3E(Q^3) = 2\mu(6,0)-(8\rho^6+12\rho^2)\mu(3,3)-12\rho^2\mu(5,1)+(24\rho^4+6)\mu(4,2).$$

By plugging in the moments from Section 2, we have the following:

$$(2m)^3[(1-\rho^2)(1+\rho^2)]^3E(Q^3)$$
$$=2(120m^3+2080m^2+3840m)$$
$$-(8\rho^6+12\rho^2)\left[(48\rho^6+72\rho^2)m^3+(288\rho^6+1152\rho^4+576\rho^2+64)m^2+(384\rho^6+2304\rho^4+1152\rho^2)m\right]$$
$$-12\rho^2(120\rho^2m^3+2080\rho^2m^2+3840\rho^2m)$$
$$+(24\rho^4+6)(96\rho^4m^3+1344\rho^4m^2+2304\rho^4m+2176\rho^2m^2+1536\rho^2m+24m^3+96m^2),$$

which can be simplified as

$$(2m)^3[(1-\rho^2)(1+\rho^2)]^3E(Q^3)$$
$$= (-384\rho^{12}+1152\rho^8-1152\rho^4+384)m^3$$
$$+(-2304\rho^{12}-9216\rho^{10}+24192\rho^8+1024\rho^6-21504\rho^4+3072\rho^2+4736)m^2$$
$$+(-3072\rho^{12}-18432\rho^{10}+41472\rho^8+9216\rho^6-46080\rho^4+9216\rho^2+7680)m,$$

or,

$$m^2[(1-\rho^2)(1+\rho^2)]^3E(Q^3)$$
$$= (-48\rho^{12}+144\rho^8-144\rho^4+48)m^3$$
$$+(-288\rho^{12}-1152\rho^{10}+3024\rho^8+128\rho^6-2688\rho^4+384\rho^2+592)m^2$$
$$+(-384\rho^{12}-2304\rho^{10}+5184\rho^8+1152\rho^6-5760\rho^4+1152\rho^2+960)m.$$

Dividing both sides by $(1-\rho^2)^3$, we have (*ii*).

Corollary 4.1: Let U and V have the bivariate chi-square distribution with density function in Theorem 1.1, and Q be defined by (4.1). Then for $m>2$ and $\rho=0$, the second and third order standardized moments of the bivariate chi-square distribution with density function in (1.1) is given by:

$(i)\qquad E(Q^2) = 8 + (24/m),$

$(ii)\qquad E(Q^3) = 48 + (592/m) + (960/m^2).$

Indeed the above are the second and the third order standardized moments of the bivariate chi-square distribution with density function given by (1.2). The corollary can also be proved by using (2.2) in Corollary 3.1. These moments also coincide, as expected, to that of the bivariate normal distribution as m tends to infinity.

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