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Abstract The standardized moments or Mahalanobis moments are easily calculated for bivariate elliptical distribution which includes bivariate normal as a special case. But for many other bivariate distributions these are challenging. A set of alternative formulae is developed to derive Mahalanobis moments for any bivariate distribution. The second order Mahalanobis moment accounts for the coefficient of kurtosis. The proposed method works well if the product moments have closed forms. Ideas are illustrated with examples.

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1. Introduction

In a series of papers, Mardia (1970; 1974; 1975) defined and discussed the properties of measures of kurtosis and skewness based on Mahalanobis distance. The coefficient of kurtosis is the second order moment of standardized distance better known as Mahalanobis distance. The moments can be referred to as standardized moments, Mahalanobis moments or Mardia moments. Interested readers may go through Kotz, Balakrishnan and Johnson (2000) for an excellent discussion on multivariate skewness and kurtosis.

For some distributions, it is easy to derive the distribution of the Mahalanobis distance and calculate moments but for others these are difficult. Kotz, Balakrishnan and Johnson (2000) have provided the kurtosis of the Marshall-Olkin bivariate exponential distribution. In this paper we provide an alternative method to calculate Mahalanobis moments for bivariate distributions in terms of product moments of the components of a bivariate vector. Product moments (also called raw product moments or product moments around zero) of order a and b for two random variables X_1 and X_2 are defined by $\mu'(a,b) = E\left(X_1^a X_2^b\right)$ while the centered product moments (sometimes called central product moments, corrected product moments or central mixed moments) are defined by

$$\mu(a,b) = E\left[\left(X_1 - E(X_1)\right)^a \left(X_2 - E(X_2)\right)^b\right]. \quad (1.1)$$

Interested readers may go through Johnson, Kotz and Kemp (1993, 46) or Johnson, Kotz and Balakrishnan (1997, 3). Evidently $\mu'(a,0)=E\left(X_1^a\right)$ is the a-th moment of X_1, and $\mu'(0,b)=E\left(X_2^b\right)$ is the b-th moment of X_2. In case X_1 and X_2 are independent, then $\mu'(a,b)=E\left(X_1^a)E(X_2^b\right)=\mu'(a,0)\mu'(0,b)$ and $\mu(a,b)=\mu(a,0)\mu(0,b)$. The correlation coefficient $\rho(-1<\rho<1)$ between X_1 and X_2 is denoted by

$$\rho_{X_1,X_2}=\frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}. \tag{1.2}$$

Note that $\mu(2,0)=E\left(X_1-E(X_1)\right)^2=V(X_1)=\sigma_{20}$ which is popularly denoted by σ_1^2 while the central product moment, $\mu(1,1)=E\left[\left(X_1-E(X_1)\right)\left(X_2-E(X_2)\right)\right]$ denoted popularly by σ_{12}, is, in fact, the covariance between X_1 and X_2.

In this paper, we derive Mahalanobis moments in terms of centered product moments. Mahalanobis moments for a bivariate normal distribution and bivariate *t*-distribution are calculated. It is observed that general formulae for Mahalanobis moments for bivariate elliptical distribution, which includes bivariate normal and *t*-distributions as special cases, are easily obtained. An example of bivariate chi-square distribution is considered for which the proposed method developed in Section 2 seems to be appropriate. In what follows we will rather use $X_1=X$ and $X_2=Y$ to avoid all confusion of a trivial nature, and define $\mu(a,b)=E\left[(X-\xi)^a(Y-\theta)^b\right]$ where $\xi=E(X),\theta=E(Y)$.

2. Mahalanobis Moments in Terms of Product Moments

For a bivariate random vector $W=(X,Y)'$, with mean vector $\mu=(\xi,\theta)'$ and covariance matrix

$$Cov(W)=E(W-\mu)(W-\mu)'=\begin{pmatrix}\mu(2,0) & \mu(1,1)\\ \mu(1,1) & \mu(0,2)\end{pmatrix}=\Omega \text{ (say)},$$

the standardized distance is defined by

$$\begin{aligned}Q&=(W-\mu)'\Omega^{-1}(W-\mu)\\ &=((X-\xi)\ \ (Y-\theta))\Omega^{-1}((X-\xi)\ \ (Y-\theta))'.\end{aligned} \tag{2.1}$$

The quantity Q is also known to be generalized distance or Mahalanobis distance. For a bivariate random vector W with $E(W)=\mu$ and $Cov(W)=\Omega$, we define standardized moments or Mahalanobis moments by $\beta_i = E(Q^i),\ i=1,2,\cdots$ where $Q=(W-\mu)'\Omega^{-1}(W-\mu)=\|\Omega^{-1/2}(W-\mu)\|^2$. Of special interest is the the second order Mahalanobis moments by $\beta_2 = E(Q^2)$ which is the coefficient of kurtosis in the sense of Mardia (1979c). For a recent paper on kurtosis, see An and Ahmed (2007), and the references therein.

Note that Kotz, Nadarajah and Mitov (2003) presented an elegant technique for product moments of the components of any multivariate random vectors in terms of cumulative distribution function or survival function. It appears that if the cumulative distribution function or the survival function has a closed form, the Nadarajah and Mitov (2003) technique works well. For Marshall-Olkin bivariate exponential distribution with survival function

$$P(X \ge x, Y \ge y) = \begin{cases} e^{-x-(1+\lambda)y}, 0 \le x \le y \\ e^{-y-(1+\lambda)x}, 0 \le y \le x \end{cases}$$

where $\lambda > 0$, Nadarajah and Mitov (2003) calculated raw product moment of general order from which it is possible to calculate Mahalanobis moments of the distribution. Kotz, Balakrishnan and Johnson (2000, 82) mentioned that the coefficient of kurtosis of the distribution is given by $\beta_2 = 2(1+\rho)^{-3}(3\rho^4+9\rho^3+15\rho^2+12\rho+4)$ where the correlation coefficient ρ is given by $(\lambda+2)\rho=\lambda$. They also mentioned that in case $\rho=0$, the components X and Y become independent, in which case $\beta_2 = 8$ (which is the same as that of the bivariate normal distribution). Interested readers may go through Kotz, Nadarajah and Mitov (2003) for a useful formula for product moments for any univariate distribution.

In fact regardless of the distribution of the variable in question the first order standardized moment is the dimension of the random variable. Let W be a p-component random vector with $E(W)=\mu$ and $Cov(W)=\Omega$, then

$$\beta_1 = E(trW'\Omega^{-1}W) = E(tr\Omega^{-1}WW') = tr[\Omega^{-1}E(WW')] = tr(\Omega^{-1}\Omega) = tr(I_p) = p. \qquad (2.2)$$

If $W \sim N_p(\mu,\Omega)$, then $Q = (W-\mu)'\Omega^{-1}(W-\mu) \sim \chi_p^2$ so that $\beta_k = E(Q^k)$, the standardized moment of order k follows from chi-square distribution. In case W has a multivariate t-distribution or multivariate elliptical distribution, standardized moments can also be calculated without much difficulty. In this paper we developed general formulae for standardized moments for any bivariate distribution. We also provided some examples to illustrate the idea. We remark that in case W has a complicated distribution, say a bivariate chi-square distribution (Theorem 4.4), it would be much difficult to calculate the distribution of Q and hence the standardized moments. In the following theorem, we derive a set of formulae for standardized moments for any bivariate distribution in terms of centered product moments just to demonstrate the potential of an alternative way.

Theorem 2.1 Let $\mu(a,b)$ be centered product moments between X and Y. Then

$$
\begin{aligned}
(i)\ & \left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^2 E(Q^2) \\
&= \mu(4,0)\mu^2(0,2)+\mu(0,4)\mu^2(2,0) \\
&+4\mu^2(1,1)\mu(2,2)+2\mu(2,0)\mu(0,2)\mu(2,2) \\
&-4\mu(0,2)\mu(1,1)\mu(3,1)-4\mu(2,0)\mu(1,1)\mu(1,3),
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ & \left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^3 E(Q^3) \\
&= \mu(6,0)\mu^3(0,2)+\mu^3(2,0)\mu(0,6) \\
&-6\mu^2(0,2)\mu(1,1)\mu(5,1)-6\mu^2(2,0)\mu(1,1)\mu(1,5) \\
&+12\mu(0,2)\mu^2(1,1)\mu(4,2)+12\mu(2,0)\mu^2(1,1)\mu(2,4) \\
&+3\mu(2,0)\mu^2(0,2)\mu(4,2)+3\mu^2(2,0)\mu(0,2)\mu(2,4) \\
&-8\mu^3(1,1)\mu(3,3)-12\mu(2,0)\mu(0,2)\mu(1,1)\mu(3,3).
\end{aligned}
$$

Proof. From (2.1) we have

$$
Q = (X-\xi \quad Y-\theta)\begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix}^{-1} \begin{pmatrix} X-\xi \\ Y-\theta \end{pmatrix},
$$

which can be simplified as

$$
\begin{aligned}
&[\mu(2,0)\mu(0,2)-\mu^2(1,1)]Q \\
&= \mu(0,2)(X-\xi)^2 - 2\mu(1,1)(X-\xi)(Y-\theta)+\mu(2,0)(Y-\theta)^2.
\end{aligned} \tag{2.4}
$$

By taking expected values in both sides of the above identity, we have

$$[\mu(2,0)\mu(0,2)-\mu^2(1,1)]E(Q)$$
$$=\mu(2,0)\mu(0,2)-2\mu^2(1,1)+\mu(0,2)\mu(2,0)$$
$$=\mu(2,0)\mu(0,2)-2\rho^2\mu(2,0)\mu(0,2)+\mu(0,2)\mu(2,0)$$
$$=2\mu(2,0)\mu(0,2)(1-\rho^2),$$

i.e. $E(Q)=2$

which is generally true (see 2.2). By squaring both sides of (2.4) we have

$$\left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^2 Q^2$$
$$=\mu^2(0,2)(X-\xi)^4+4\mu^2(1,1)(X-\xi)^2(Y-\theta)^2+\mu^2(2,0)(Y-\theta)^4-4\mu(0,2)\mu(1,1)(X-\xi)^3(Y-\theta)$$
$$+2\mu(2,0)\mu(0,2)(X-\xi)^2(Y-\theta)^2-4\mu(2,0)\mu(1,1)(X-\xi)(Y-\theta)^3.$$

Then the result in (i) follows by taking expected values in both sides of the above identity.
By cubing both sides of (2.4) we have:

$$\left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^3 Q^3$$
$$=\mu^3(0,2)(X-\xi)^6-8\mu^3(1,1)(X-\xi)^3(Y-\theta)^3+\mu^3(2,0)(Y-\theta)^6$$
$$-6\mu^2(0,2)\mu(1,1)(X-\xi)^5(Y-\theta)+12\mu(0,2)\mu^2(1,1)(X-\xi)^4(Y-\theta)^2$$
$$+3\mu(2,0)\mu^2(0,2)(X-\xi)^4(Y-\theta)^2+3\mu^2(2,0)\mu(0,2)(X-\xi)^2(Y-\theta)^4$$
$$+12\mu(2,0)\mu^2(1,1)(X-\xi)^2(Y-\theta)^2-6\mu^2(2,0)\mu(1,1)(X-\xi)(Y-\theta)^5$$
$$-12\mu(2,0)\mu(0,2)\mu(1,1)X-\xi)^3(Y-\theta)^3.$$

Part (ii) follows by taking expected values of the above identity.

Corollary 2.1 Let $\mu(a,b)$ be the centered product moment and $\rho=(\mu(2,0)\mu(0,2))^{-1/2}\mu(1,1)$ be the correlation coefficient between X and Y. Then

$$(i)\left[\mu(2,0)\mu(0,2)(1-\rho^2)\right]^2 E(Q^2)$$
$$=\mu(4,0)\mu^2(0,2)+\mu(0,4)\mu^2(2,0)+(4\rho^2+2)\mu(2,0)\mu(0,2)\mu(2,2)$$
$$-4\rho\left(\mu(2,0)\mu(0,2)\right)^{1/2}[\mu(0,2)\mu(3,1)+\mu(2,0)\mu(1,3)],$$

$$(ii)\left[\mu(2,0)\mu(0,2)(1-\rho^2)\right]^3 E(Q^3)$$
$$=\mu(6,0)\mu^3(0,2)+\mu^3(2,0)\mu(0,6)-\left(\mu(2,0)\mu(0,2)\right)^{3/2}\mu(3,3)4\rho(2\rho^2+3)$$
$$-6\left(\mu(2,0)\mu(0,2)\right)^{1/2}\rho[\mu^2(0,2)\mu(5,1)+\mu^2(2,0)\mu(1,5)]$$
$$+3\mu(2,0)\mu(0,2)(4\rho^2+1)\left[\mu(0,2)\mu(4,2)+\mu(2,0)\mu(2,4)\right].$$

Corollary 2.2 Let X and Y have a bivariate distribution with $E(X^aY^b)=E(X^bY^a)$ and correlation coefficient ρ. Then

$$(i)\ \mu^2(0,2)(1-\rho^2)^2E(Q^2)=2\mu(4,0)+(4\rho^2+2)\mu(2,2)-8\rho\mu(3,1),$$
$$(ii)\ \mu^3(0,2)(1-\rho^2)^3E(Q^3)=2\mu(6,0)-(8\rho^3+12\rho)\mu(3,3)-12\rho\mu(5,1)+(24\rho^2+6)\mu(4,2).$$

Corollary 2.3 Let X and Y have a bivariate distribution. If X and Y are independent, then

$$(i)\ E(Q^2)=2+\frac{\mu(4,0)}{\mu^2(2,0)}+\frac{\mu(0,4)}{\mu^2(0,2)},$$
$$(ii)\ E(Q^3)=\frac{\mu(6,0)}{\mu^3(2,0)}+\frac{\mu(0,6)}{\mu^3(0,2)}+3\left(\frac{\mu(4,0)}{\mu^2(2,0)}+\frac{\mu(0,4)}{\mu^2(0,2)}\right).$$

Corollary 2.4 Let X and Y have a bivariate distribution. If X and Y are independently and identically distributed, then

$$(i)\ E(Q^2)=2\left(1+\frac{\mu(4,0)}{\mu^2(2,0)}\right),$$
$$(ii)\ E(Q^3)=2\left(\frac{\mu(6,0)}{\mu^3(2,0)}+3\frac{\mu(4,0)}{\mu^2(2,0)}\right).$$

3. Centered Product Moments of the Bivariate Normal Distribution

The pdf (probability density function) of the bivariate normal distribution is given by

$$f_1(x,y)=\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\ \exp\left(\frac{-q(x,y)}{2}\right), \tag{3.1}$$

where

$$(1-\rho^2)q(x,y)=\left(\frac{x-\xi}{\sigma_1}\right)^2+\left(\frac{y-\theta}{\sigma_2}\right)^2-\frac{2\rho(x-\xi)(y-\theta)}{\sigma_1\sigma_2}.$$

The following theorem is due to Kendal and Stuart (1969, 91).

Theorem 3.1 The centered product moments $\mu(a,b)=E\left[(X-\xi)^a(Y-\theta)^b\right]$ of the bivariate normal distribution with pdf in (3.1) are given by

$\mu(a,b)=\sigma_1^a\sigma_2^b\lambda(a,b)$ where

$\lambda(a,b)=(a+b-1)\rho\lambda(a-1,b-1)+(a-1)(b-1)(1-\rho^2)\lambda(a-2,b-2),$

$$\lambda(2a,2b) = \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$
$$\lambda(2a+1,2b+1) = \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$
$$\lambda(2a,2b+1) = \lambda(2a+1,2b) = 0.$$

The above can be rewritten as

$$\mu(a,b) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1) + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2),$$
$$\mu(2a,2b) = \sigma_1^{2a}\sigma_2^{2b} \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$
$$\mu(2a+1,2b+1) = \sigma_1^{2a+1}\sigma_2^{2b+1} \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$
$$\mu(2a,2b+1) = \mu(2a+1,2b) = 0.$$

Product moments that are needed for deriving standardized moments up to order 3 are provided below:

$$\mu(2,0) = \sigma_1^2,$$
$$\mu(0,2) = \sigma_2^2,$$
$$\mu(4,0) = 3\sigma_1^4,$$
$$\mu(0,4) = 3\sigma_2^4,$$
$$\mu(6,0) = 15\sigma_1^6,$$
$$\mu(0,6) = 3\sigma_1^6,$$

$$\mu(1,1) = \rho\sigma_1\sigma_2,$$
$$\mu(2,2) = (1+2\rho^2)\sigma_1^2\sigma_2^2,$$
$$\mu(1,3) = 3\rho\sigma_1\sigma_2^3,$$
$$\mu(3,1) = 3\rho\sigma_1^3\sigma_2,$$

$$\mu(2,4) = 3(1+4\rho^2)\sigma_1^2\sigma_2^4,$$
$$\mu(3,3) = 3\rho(3+2\rho^2)\sigma_1^3\sigma_2^3,$$
$$\mu(4,2) = 3(1+4\rho^2)\sigma_1^4\sigma_2^2,$$
$$\mu(1,5) = 15\rho\sigma_1\sigma_2^5,$$
$$\mu(5,1) = 15\rho\sigma_1^5\sigma_2.$$

4. Some Examples

(i) Bivariate Normal Distribution

Let us represent the bivariate normal distribution with pdf in (3.1) by,

$$W = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \Sigma),\ \mu = \begin{pmatrix} \xi \\ \theta \end{pmatrix}, \Sigma = \begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix}.$$

It is known that for a p-variate normal distribution, $W \sim N_p(\mu, \Sigma)$, the standardized distance $Q = (W - \mu)'\Sigma^{-1}(W - \mu) \sim \chi_p^2$ so that $\beta_1 = E(Q) = p$, $\beta_2 = E(Q^2) = p(p+2)$ and $\beta_3 = E(Q^3) = p(p+2)(p+4)$. That is for the univariate normal distribution, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 15$ and for the bivariate normal distribution,

$$\beta_1 = 2, \beta_2 = 8, \beta_3 = 48. \tag{4.1}$$

We derive Mahalanobis moments for bivariate normal and bivariate *t*-distribution by the method developed in Section 2. It may be mentioned that the standardized moments for bivariate elliptical distributions are easily obtained, but these are not easy for other distributions as it is difficult to derive the distribution of the standardized distance.

Theorem 4.1 The second and the third order standardized moments of bivariate normal distribution are given by $\beta_2 = 8, \beta_3 = 48$.

Proof. By the use of moments from Section 3, it follows from Theorem 2.1 (i) or preferably Corollary 2.1(i) that

$$\begin{aligned}
&\left[\sigma_1^2\sigma_2^2(1-\rho^2)\right]^2 E(Q^2) \\
&= (3\sigma_1^4)(\sigma_2^2)^2 + (3\sigma_2^4)(\sigma_1^2)^2 + 4(\rho^2\sigma_1^2\sigma_2^2)[\sigma_1^2\sigma_2^2(1+2\rho^2)] \\
&- 4(\sigma_2^2)(\rho\sigma_1\sigma_2)(3\sigma_1^3\sigma_2\rho) - 4\sigma_1^2(\rho\sigma_1\sigma_2)(3\sigma_1\sigma_2^3\rho), \\
&+ 2\sigma_1^2\sigma_2^2[\sigma_1^2\sigma_2^2(1+2\rho^2)]
\end{aligned}$$

so that

$$\begin{aligned}
(1-\rho^2)^2 E(Q^2) &= 3 + 3 + 4\rho^2(1+2\rho^2) + 2(1+2\rho^2) - 4\rho(3\rho) - 4\rho(3\rho) \\
&= 8(1 - 2\rho^2 + \rho^4).
\end{aligned}$$

Similarly by plugging in the moments from Section 3, it follows from Theorem 2.1 (ii) or preferably Corollary 2.1(ii) that

$$\left[\sigma_1^2\sigma_2^2(1-\rho^2)\right]^3 E(Q^3)$$
$$= (15\sigma_1^6)(\sigma_2^2)^3 + (\sigma_1^2)^3(15\sigma_2^6)$$
$$-8(\rho^3\sigma_1^3\sigma_2^3)[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)] - 12\sigma_1^2\sigma_2^2(\rho\sigma_1\sigma_2)[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)]$$
$$-6\sigma_2^4(\rho\sigma_1\sigma_2)[15\sigma_1^5\sigma_2\rho] - 6\sigma_1^4(\rho\sigma_1\sigma_2)(15\sigma_1\sigma_2^5\rho)$$
$$+12\sigma_2^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^4\sigma_2^2(1+4\rho^2)] + 12\sigma_1^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)]$$
$$+3\sigma_1^2(\sigma_2^2)^2[3\sigma_1^4\sigma_2^2(1+4\rho^2)] + 3(\sigma_1^2)^2(\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)]$$

so that

$$(1-\rho^2)^3 E(Q^3)$$
$$= 15 + 15 - 8\rho^3[3\rho(3+2\rho^2)] - 12\rho[3\rho(3+2\rho^2)]$$
$$-6\rho(15\rho) - 6\rho(15\rho) + 12\rho^2[3(1+4\rho^2)] + 12\rho^2[3(1+4\rho^2)]$$
$$+3[3(1+4\rho^2)] + 3[3(1+4\rho^2)]$$
$$= 48(1-3\rho^2+3\rho^4-\rho^6).$$

(ii) **Bivariate T-Distribution**

Let $X' = (X_1, X_2)$ be the bivariate t-random vector with pdf

$$f_2(x) = (2\pi)^{-1} |\Sigma|^{-1/2} \left(1 + (x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} \tag{4.1}$$

where $\theta' = (\theta_1, \theta_2)$ is an unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters while the scalar ν is assumed to be a known positive constant (Anderson, 2003, 289). For recent update on t-distributions see Kotz and Nadarajah (2005) and Kibria (2006) and the references therein.

The following theorem, due to Joarder (2006a), is needed to calculate Mahalanobis moments of the above bivariate *t*-distribution given by (4.1),

Theorem 4.2 The centered product moments of the bivariate *t*-distribution with pdf in (4.1) are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)\gamma_4,$$

$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$$

$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1}\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$$

$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0$$

where $\gamma_a = \dfrac{(\nu/2)^{a/2}\Gamma(\nu/2 - a/2)}{\Gamma(\nu/2)}, \quad \nu > a$

By the use of the above moments in Theorem 2.1 or preferably in Corollary 2.1, we have the second and third order Mahalanobis moments of the bivariate *t*-distribution having pdf in (4.1):

$$\beta_2 = 8\frac{\nu-2}{\nu-4}, \; \nu > 4,$$

$$\beta_3 = 48\frac{(\nu-2)^2}{(\nu-4)(\nu-6)}, \; \nu > 6.$$

(iii) **Multivariate Elliptical Distribution**

The second and third order Mahalanobis moments are calculated for p-variate elliptical distribution. With $p = 2$, the results boil down to bivariate elliptical distribution. Consider the multivariate elliptical distribution with pdf

$$f_3(x) = g((x-\mu)'\Sigma^{-1}(x-\mu)), \tag{4.2}$$

where x is a p-dmensional column vector with mean $E(X) = \mu$ and the covariance matrix $Cov(X) = p^{-1}E(R^2)\Sigma$ where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X-\mu)$. Then we have the following theorem (cf. Anderson, 2003, 103):

Theorem 4.3 Let X have the multivariate elliptical distribution with pdf in (4.2). Then the second and the third order Mahalanobis moments of the distribution are given by

$$\beta_2 = E(Q^2) = p^2\frac{E(R^4)}{E^2(R^2)}, \text{ and}$$

$$\beta_3 = E(Q^3) = p^3\frac{E(R^6)}{E^3(R^2)}$$

respectively, where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X-\mu)$.

Proof. The covariance matrix of the elliptical distribution is given by $Cov(X) = p^{-1}E(R^2)\Sigma$ so that the standardized distance is given by $Q = (X-\mu)'(p^{-1}E(R^2)\Sigma)^{-1}(X-\mu)$. Then the theorem is obvious by virtue of

$$Q = \frac{pR^2}{E(R^2)}, \; Q^2 = \frac{p^2R^4}{E^2(R^2)}, \; Q^3 = \frac{p^3R^6}{E^3(R^2)}.$$

Note that if the form of $g(.)$ is known, the second and the third order Mahalanobis moments of the distribution can be calculated by the pdf of R given by

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, g(r^2),\ 0 < r.$$

It is well known that for the multivariate normal distribution $R^2 \sim \chi^2_p$, and for the multivariate *t*-distribution with pdf

$$f_4(x) = \frac{\Gamma((\nu+p)/2)}{\Gamma(\nu/2)(\nu\pi)^{p/2}} |\Sigma|^{-1/2} \left(1 + (x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1},\ \ \nu > 2,$$

we have $p^{-1}R^2 \sim F(p,\nu)$.

(iv) Bivariate Chi-square Distribution

The following bivariate chi-square distribution is derived by Joarder (2007a).

Theorem 4.4 The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its pdf is given by

$$f_5(u,v) = \frac{(uv)^{m/2-1}\, e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m \sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}} \sum_{k=0}^{\infty} [1+(-1)^k] \left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k \frac{\Gamma\left((k+1)/2\right)}{k!\,\Gamma\left((k+m)/2\right)}$$

$m > 2$, $-1 < \rho < 1$.

Since it is difficult to derive the distribution of Mahalanobis distance of the above bivariate chi-square distribution, Mahalanobis moments of the above bivariate chi-square distribution can be derived by using the results developed in Section 2. In case $\rho = 0$, the pdf of the joint probability distribution in Theorem 4.4, would be that of the product of two independent chi-square random variables $U \sim \chi^2_m$ and $V \sim \chi^2_m$.

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