# Algebraic Inequalities for Measures of Dispersion[1]

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ABSTRACT Some upper and lower bounds for sample standard deviation are established in terms of sample mean, median, range, the smallest and the largest order statistics. Upper bounds for variance are also derived for odd and even sample sizes whenever the sample observations are of the same sign. They are used to find bounds for some well-known sample statistics: z-scores, coefficient of variation, coefficient of skewness and the least squares estimator of the slope parameter in the context of a simple linear regression. Statistical inference of related parameters can be improved on the basis of these fixed sample properties.

***Keywords:*** Inequalities in statistics; sample mean; sample median; standard deviation; $z$-score, coefficient of variation; coefficient of skewness; regression parameters.

## 1. Introduction

Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$. For $0<p<1$, the $p$ th quantile $x_p$ of $X$ is defined by $P(X \le x_p) \ge p$ and $P(X \ge x_p) \ge 1-p$ or equivalently $P(X < x_p) \le p \le P(X \le x_p)$ (Rohatji, 1984, 164). For example if $p = 1/2$, then $x_p = \tilde{\mu}$, the median of the random variable $X$. Page and Murty (1982 and 1983) published an elementary proof of the inequality $|\tilde{\mu}-\mu| \le \sigma$. O'Cinneide (1990) presented a new proof for $|\tilde{\mu}-\mu| \le \sigma$ and stated the following generalization.

**Proposition 1.1** Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$. Then for $0<p<1$ and $q=1-p$, the following inequality holds

$|x_p-\mu| \le \sigma \max\left(\sqrt{p/q}, \sqrt{q/p}\right)$ where $x_p$ is the $p$ th quantile.

For $p = 1/2$, it follows from the above proposition that $|\tilde{\mu}-\mu| \le \sigma$. Dharmadhikari (1991) noted that for $p \ne 1/2$, the inequality is somewhat unsatisfactory. The refined inequality proved by her with the help of one-sided Chebyshev inequality is stated in the following theorem.

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**Proposition 1.2** Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$. Then for $0 < p < 1$ and $q = 1 - p$, the following inequalities hold

$$\mu - \sigma\sqrt{q/p} \le x_p \le \mu + \sigma\sqrt{q/p} .$$

For stimulating discussions, readers may go through Mallows (1991) and the references therein. A more general inequality than that in Proposition 1.2 relating sample standard deviation to mean and the $i$-th order statistic discussed by David (1988 and 1991) is presented in Theorem 1.2. Interested readers can go through the references in David (1988) for bounds of order statistics.

Sample standard deviation $(s)$ or variance $(s^2)$ is nonnegative, and is defined by $(n-1)s^2 = \sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$. But for most data sets, the range of $s$ is much narrower than the nonnegative part of the real line. This motivated us to find bounds for standard deviation and related statistics. Some representations of sample variance are discussed in Joarder (2002). Further it has been proved by Joarder (2003) that if a computer program is used to calculate sample variance, then it can be efficiently calculated by the representation based on the first order differences of observations. Another motivation for the current research is the improved inference in situations when the parameter is known to have a restricted space (Silvapulle and Sen, 2004).

It is well known that $\sqrt{n-1}\, s \ge \sqrt{n}\, u$ where $u$ is the mean absolute deviation of sample values around the mean defined by $nu = \sum_{i=1}^{n} |x_i - \bar{x}|$. Let $n$ ordered sample observations be denoted by $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$. It is also well known that for $n = 2$, the sample standard deviation has a simpler form given by $s = w/\sqrt{2}$ where $w = x_{(n)} - x_{(1)}$ is the sample range. Shiffler and Harsha (1980) have formulated an upper bound for the sample standard deviation $(s)$ in terms of the sample range $w$, while Mcleod and Henderson (1984) have determined a lower bound for $s$ in terms of $w$ which also follows from Thomson (1955). Eisenhauer (1993) combined them. A stronger version of these results with more transparent arguments is provided in Theorem 2.1.

**Theorem 1.1** (Macleod and Henderson (1984) and (Shiffler and Harsha (1980)). Let $w$ and $s$ denote, respectively, the range and standard deviation of a sample of size $n \ge 2$. Then

$$\frac{w}{\sqrt{2(n-1)}} \le s \le \frac{w}{2}\sqrt{\frac{n}{n-1}} .$$

**Theorem 1.2** (David, 1991) For $1 \le i \le n$, let $x_{(i)}$ be the $i$th order statistic and $s$, the standard deviation based on a sample of size $n \ge 2$. Then

$$|x_{(i)} - \bar{x}| \le s \max\left( \sqrt{\frac{(n-1)(i-1)}{n(n+1-i)}},\ \sqrt{\frac{(n-1)(n-i)}{ni}} \right).$$

By the use of Theorems 1.1 and 1.2 we immediately obtain the following corollaries:

**Corollary 1.1** For $1 \le i \le n$, let $x_{(i)}$ be the $i$th order statistic from a sample of size $n \ge 2$. Then

$$\sqrt{\frac{n}{\max(i-1, n-i)} - 1}\ \sqrt{\frac{n}{n-1}}\ |x_{(i)} - \bar{x}| \le s \le \frac{w}{2}\sqrt{\frac{n}{n-1}}.$$

**Corollary 1.2** (Eisenhauer, 1993) Let $w$ and $s$ denote the range and standard deviation of a sample of size $n$. Then

(a) $\dfrac{1}{\sqrt{2(n-1)}} \le \dfrac{s}{w} \le \dfrac{1}{2}\sqrt{\dfrac{n}{n-1}}$,

(b) $0 \le s/w \le 1/2$ as $n \to \infty$.

## 2. Some Inequalities in Descriptive Statistics

The following result is a refined version of Theorem 1.1.

**Theorem 2.1** Let $\bar{x}, \tilde{x}, w$ and $s$ respectively denote the mean, median, range and standard deviation of a sample of size $n$. Then

$$\frac{w}{\sqrt{2\ (n-1)}} \le \sqrt{\frac{w^2}{2(n-1)} + \frac{(\tilde{x} - \bar{x})^2}{2}} \le s \le \sqrt{\frac{n(\bar{x} - x_{(1)})(x_{(n)} - \bar{x})}{n-1}} \le \frac{w}{2}\sqrt{\frac{n}{n-1}}.$$

**Proof.** For any $a$ and $n \ge 2$, $(n-1)s^2 = \sum_{i=1}^{n}(x_i - a)^2 - \frac{1}{n}\left(\sum_{i=1}^{n}(x_i - a)\right)^2$ so that for the ordered sample observations $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, we have $(n-1)s^2 \le n\max\left((x_{(1)} - a)^2, (x_{(n)} - a)^2\right) - n(\bar{x} - a)^2$. In particular, for $a = (x_{(1)} + x_{(n)})/2 = x_{(1)} + w/2$, we have

$$(n-1)s^2 \le nw^2/4 - n\left((\bar{x} - x_{(1)}) - w/2\right)^2 = n(\bar{x} - x_{(1)})(x_{(n)} - \bar{x}). \tag{2.1}$$

Since $ab \le (a+b)^2/4$ for any real numbers $a$ and $b$, we deduce by putting $a = \bar{x} - x_{(1)} \ge 0$, $b = x_{(n)} - \bar{x} \ge 0$ that $(\bar{x} - x_{(1)})(x_{(n)} - \bar{x}) \le w^2/4$. Then the two inequalities in the theorem are evident by (2.1). Next, by using $2(a^2 + b^2) \ge (a-b)^2$, for any $y_1 \le y_2 \le \cdots \le y_n$ with $w_y = y_n - y_1$, we have

$$2\sum_{i=1}^{n} y_i^2 = 2\left(y_n^2 + y_1^2\right) + 2\left(y_2^2 + \cdots + y_{n-1}^2\right) \ge w_y^2 + 2\left(y_2^2 + \cdots + y_{n-1}^2\right). \tag{2.2}$$

For $n$ odd, rewrite (2.2) as

$$2\sum_{i=1}^{n} y_i^2 = w_y^2 + 2\left(y_2^2 + \cdots + y_{(n+1)/2}^2\right) + 2\left(y_{(n+3)/2}^2 + \cdots + y_{n-1}^2\right). \quad (2.3)$$

Let $\tilde{y}$ be the median of the $y_i\,(i=1,2,\cdots,n)$ values. If $\tilde{y}<0$ then for $n\geq 3$ and $2\leq i\leq (n+1)/2$ , we have $y_i \leq \tilde{y},$ or $y_i^2 \geq \tilde{y}^2,$ so it follows from (2.3) that $2\sum_{i=1}^{n} y_i^2 \geq w_y^2 + 2\left((n+1)/2-2+1\right)\tilde{y}^2 = w_y^2 + (n-1)\tilde{y}^2$. If $\tilde{y}\geq 0$ then for $n\geq 3$ and $(n+1)/2)\leq i\leq (n-1)$, we have $\tilde{y} \leq y_i$, so that it follows from (2.3) that $2\sum_{i=1}^{n} y_i^2 \geq w_y^2 + 2(n-1-(n+1)/2+1)\tilde{y}^2 = w_y^2 + (n-1)\tilde{y}^2$.

Next, let $n$ be even. For $n=2$ we have $2\left(y_1^2+y_2^2\right)=(y_2-y_1)^2+(y_2+y_1)^2=w_y^2+(2\tilde{y})^2\geq w_y^2+2\tilde{y}^2$. For $n=4$, we have $2\sum_{i=1}^{4} y_i^2=(y_4-y_1)^2+(y_3-y_2)^2+(y_4+y_1)^2+(y_3+y_2)^2\geq w_y^2+4\tilde{y}^2$. Since $2(a^2+b^2)\geq (a+b)^2$, rewrite (2.2) for even $n\geq 2$ as

$$2\sum_{i=1}^{n} y_i^2 \geq w_y^2 + 2\left(y_2^2 + \cdots + y_{n/2-1}^2\right) + \left(y_{n/2} + y_{n/2+1}\right)^2 + 2\left(y_{n/2+2}^2 + \cdots + y_{n-1}^2\right). \quad (2.4)$$

If $\tilde{y}<0$ , then for even $n\geq 6$ and $n/2\leq i\leq n/2-1$, we have $y_i\leq\tilde{y},$ i.e. $y_i^2\geq\tilde{y}^2,$ and it follows from (2.4) that $2\sum_{i=1}^{n} y_i^2 \geq w_y^2 + 2(n/2-1-2+1)\tilde{y}^2 + (2\tilde{y})^2 \geq w_y^2 + n\tilde{y}^2$. Similarly, if $\tilde{y}\geq 0$ , then for even $n\geq 6$ and $n/2{+}2\leq i\leq n-1$, we have $y_i\leq\tilde{y},$ and it follows from (2.4) that $2\sum_{i=1}^{n} y_i^2 \geq w_y^2 + 2\left(y_2^2+\cdots+y_{n/2-1}^2\right)+\left(2\tilde{y}\right)^2+2\left(n-1-(n/2+2)+1)\tilde{y}^2\right)\geq w_y^2+n\tilde{y}^2$. So for even $n\geq 2$, $2\sum_{i=1}^{n} y_i^2 \geq w^2 + n\tilde{y}^2$. In all cases for $n\geq 2$, we have

$$(n-1)s^2 \geq \begin{cases} w^2 + (n-1)\tilde{y}^2 & \text{if } n \text{ is odd} \\ w^2 + n\tilde{y}^2 & \text{if } n \text{ is even} \end{cases} \quad (2.5)$$

By putting $y_i = x_i - \bar{x},\ (1\leq i\leq n),\ n\geq 2$ in (2.5), we have

$$2(n-1)s^2 \geq \begin{cases} w^2 + (n-1)(\tilde{x}-\bar{x})^2 & \text{if } n \text{ is odd} \\ w^2 + n(\tilde{x}-\bar{x})^2 & \text{if } n \text{ is even} \end{cases}$$

as required.

By putting $y_i = x_i - f(\underset{\sim}{x}),\ (1\leq i\leq n),\ n\geq 2$, in (2.5) we have $2(n-1)s^2 \geq w^2 + (n-1)\left(\bar{x} - f(\underset{\sim}{x})\right)^2$

where $f(\underset{\sim}{x})$ is a function of sample values $\underset{\sim}{x}$ say the geometric mean $g(\underset{\sim}{x})$ or harmonic mean $h(\underset{\sim}{x})$. By putting $y_i = x_i - \bar{x} - g(\underset{\sim}{x}) - h(\underset{\sim}{x}),\ (1 \le i \le n),\ n \ge 2$, in (2.5) we have
$2(n-1)s^2 \ge w^2 + (n-1)\left[\tilde{x} - \bar{x} - (g(\underset{\sim}{x}) + h(\underset{\sim}{x}))\right]^2$.

Cartwright and Field (1978) provided bounds for variance in terms of the geometric mean and the most extreme observations as follows:

$$\frac{2n}{n-1}\, x_{(1)}(\bar{x} - g(\underset{\sim}{x})) \le s^2 \le \frac{2n}{n-1} x_{(n)}\ (\bar{x} - g(\underset{\sim}{x})).$$

It may be remarked here that the two sides of the rightmost inequality in the theorem are equal in case $x_{(1)} + x_{(n)} = 2\bar{x}$, otherwise the sharper inequality $\sqrt{(\bar{x} - x_{(1)})(x_{(n)} - \bar{x})} < w/2$ holds. The following result improves the bound for $s$ described in Theorem 1.2. It is also in agreement with the known result that for any two observations $s = w/\sqrt{2}$ where $w = x_{(n)} - x_{(1)}$. In what follows let $[n]$ be the greatest integer function i.e. it is the largest integer not exceeding $n$.

**Corollary 2.1** For any sample of $n \ge 2$ observations $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,

$$s^2 \ge \frac{1}{2(n-1)} \sum_{i=1}^{[n/2]} w_i^2 = \frac{w_1^2}{2(n-1)} \text{ where } w_i = x_{(n-i+1)} - x_{(i)}\ ,\ (1 \le i \le m).$$

**Proof.** For any real numbers $y_1, y_2, \cdots, y_n$ and $m = [n/2]$ we have

$$\sum_{i=1}^{n} y_i^2 \ge \left(y_1^2 + y_n^2\right) + \left(y_2^2 + y_{n-1}^2\right) + \cdots + \left(y_m^2 + y_{n-m+1}^2\right) \ge \frac{1}{2}\left[\left(y_n - y_1\right)^2 + \cdots + \left(y_{n-m+1} - y_m\right)^2\right]$$

Letting $y_i = x_{(i)} - \bar{x}\ ,\ (i = 1, 2, \cdots, m)$ and $w_j = y_{n-j+1} - y_j\ ,\ (1 \le j \le m)$, it follows from the above inequality that $2(n-1)s^2 \ge w_1^2 + w_2^2 + \cdots + w_m^2 \ge w_1^2$.

**Theorem 2.2** For $1 \le i \le n$, let $x_{(i)}$ be the $i$ th order statistic from a sample of size $n \ge 2$. Then

$(i)\ \ |x_{(i)} - \bar{x}| \le s \dfrac{n-1}{\sqrt{n}}$ for each $i$

$(ii)\ \ |x_{(i)} - \bar{x}| \le s \dfrac{n-1}{\sqrt{n(n+1)}} \le s$ if $i = \dfrac{n+1}{2}$

$(iii)\ \ |\tilde{x} - \bar{x}| \le s\sqrt{\dfrac{n-1}{n}} < s$

**Proof.** (i) $\max\limits_{i}\ |x_{(i)} - \bar{x}| \le \max(x_{(n)} - \bar{x},\ \ \bar{x} - x_{(1)})$. But by Theorem 1.2 we have

$$x_{(n)} - \bar{x} \le s\ \max\left(\frac{n-1}{\sqrt{n}},\ 0\right) = s\ \frac{n-1}{\sqrt{n}} \text{ and } \bar{x} - x_{(1)} \le s\ \max\left(0,\ \frac{n-1}{\sqrt{n}}\right) = s\ \frac{n-1}{\sqrt{n}}.$$

(ii) Use Theorem 1.2

(iii) If $n$ is odd and $i=(n+1)/2$, then $x_{(i)}=\tilde{x}$, and $\frac{n-1}{\sqrt{n(n+1)}}\le\sqrt{\frac{n-1}{n}}$ for any $n\ge 1$, the inequality follows from *(ii)*. If $n$ even, then $\tilde{x}=\left(x_{(n/2)}+x_{(n/2+1)}\right)/2$, and the leftmost inequality in (iii) follows from $|\tilde{x}-\bar{x}| \le 2^{-1}\left|\left(x_{(n/2)}-\bar{x}\right)+\left(x_{(n/2+1)}-\bar{x}\right)\right|\le 2^{-1}\left(|x_{(n/2)}-\bar{x}|+|x_{(n/2+1)}-\bar{x}|\right)$ by virtue of Theorem 1.2.

Let the $z$-scores be defined by $z_i=(x_i-\bar{x})/s,\ i=1,2,\cdots,n$. Then it follows from $\sum_{i=1}^{n} z_i^2=n-1$ that $n\max_i\{|z_i|\}^2\ge n-1$ so that, by virtue of (i), $\min_i\{|z_i|\}\le\sqrt{\frac{n-1}{n}}\le\max_i\ \{|z_i|\}\le\frac{n-1}{\sqrt{n}}$ as in Hayes (2004). The upper bound is originally by Pearson and Chandra Shekhar (1936). Shiffler (1987 and 1988) argued that minimum and maximum achievable value for the largest positive $z$-score is $1/\sqrt{n}$ and $(n-1)/\sqrt{n}$ respectively.

The inequality in (iii) tells us that $\tilde{x}-s<\bar{x}<\tilde{x}+s$, or, $\bar{x}-s<\tilde{x}<\bar{x}+s$. That is sample mean and median lie within one standard deviation of each other. The following corollary is obvious from Theorem 2.1 and Theorem 2.2.

**Corollary 2.2** Let $\bar{x}$ and $\tilde{x}$ be the sample mean and median based on a sample of size $n\ge 2$. Then

$$\max\left(\sqrt{\frac{n}{n-1}}|\bar{x}-\tilde{x}|\ ,\ \sqrt{\frac{w^2}{2(n-1)}+\frac{(\bar{x}-\tilde{x})^2}{2}}\right)\ \le s\le\ \frac{w}{2}\sqrt{\frac{n}{n-1}}\,.$$

**Theorem 2.3** If the observations are of the same sign, then for any sample size $n\ge 2$, the following inequalities hold:

(i) $s^2\le n\bar{x}^2-\frac{n+1}{4}\tilde{x}^2$ if $n$ is odd,

(ii) $s^2\le n\bar{x}^2-\frac{n(n-2)}{4(n-1)}\tilde{x}^2$ if $n$ is even.

**Proof.** Without loss of generality we assume that all the observations are nonnegative. If $n$ is odd, there are $\binom{(n+1)/2}{2}$ products of the form $x_i x_j$ where $1\le i<j\le n$ and $x_i,x_j\ge\tilde{x}$. Then $\left(n\bar{x}\right)^2=\sum_{i=1}^{n}x_i^2+2\sum_{i<j}x_i x_j\ge\sum_{i=1}^{n}x_i^2+2\left(\frac{n^2-1}{8}\right)\tilde{x}^2$. If $n$ is even, there are $\binom{n/2}{2}$ products of the form $x_i x_j$ where $1\le i<j\le n$ and $x_i,x_j\ge\tilde{x}$. Then $\left(n\bar{x}\right)^2=\sum_{i=1}^{n}x_i^2+2\sum_{i<j}x_i x_j\ge\sum_{i=1}^{n}x_i^2+2\left(\frac{(n/2)(n/2-1)}{2}\right)\tilde{x}^2$. The rest of the proof is immediate.

The following corollary follows from Theorem 2.1 and Theorem 2.3.

**Corollary 2.3** For any sample of $n \ge 2$ observations $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, if all the observations are of the same sign. the following inequalities hold:

$$(i)\ s \le \min\left(\sqrt{\frac{n(\bar{x}-x_{(1)})(x_{(n)}-\bar{x})}{n-1}}, \sqrt{n\bar{x}^2 - \frac{n+1}{4}\tilde{x}^2}\right) \text{ for odd } n,$$

$$(ii)\ s \le \min\left(\sqrt{\frac{n(\bar{x}-x_{(1)})(x_{(n)}-\bar{x})}{n-1}}, \sqrt{n\bar{x}^2 - \frac{n(n-2)}{4(n-1)}\tilde{x}^2}\right) \text{ for even } n.$$

**Corollary 2.4** For any sample size $n \ge 2$, if all the observations are of the same sign, the following inequalities hold:

$$\frac{n}{n-1}(\bar{x}-\tilde{x})^2 \le s^2 \le n\bar{x}^2 - \frac{n(n-2)}{4(n-1)}\tilde{x}^2 \le n\bar{x}^2 \quad .$$

**Proof.** For any $n \ge 2$, it follows from Theorem 2.3 that $n(n-1)\bar{x}^2 \ge (n-1)s^2 + \frac{n(n-2)}{4}\tilde{x}^2$ i.e. $n\bar{x}^2 \ge s^2 + \frac{n(n-2)}{4(n-1)}\tilde{x}^2$. The leftmost inequality is by virtue of Theorem 2.2 (iii).

Note that if all the observations are of the same sign, a less sharper but simpler than the above inequality is given by $|\bar{x}-\tilde{x}| \le s \le \sqrt{n}\ |\bar{x}|$. The following corollary is by virtue of Theorem 2.1 and Corollary 2.4.

**Corollary 2.5** For any sample size $n \ge 2$, if all the observations are of the same sign**,** the following inequalities hold:

$$\max\left(\sqrt{\frac{w^2}{2(n-1)} + \frac{(\tilde{x}-\bar{x})^2}{2}},\ \sqrt{\frac{n}{n-1}}|\tilde{x}-\bar{x}|\right)$$
$$\le s \le \sqrt{\frac{n}{n-1}}\min\left(\sqrt{(n-1)\bar{x}^2 - \frac{n-2}{4}\tilde{x}^2},\ \sqrt{(\bar{x}-x_{(1)})(x_{(n)}-\bar{x})}\right).$$

The following corollary is obvious by virtue of Theorem 1.2 and Corollary 2.3.

**Corollary 2.6** For any sample of *n* nonnegative observations ($n \ge 2$), the following inequalities hold:

$$\frac{1}{\sqrt{n-1}}\max\left(\frac{w}{\sqrt{2}},\ \sqrt{n}|\bar{x}-\tilde{x}|\right) \le s \le \min\left(\sqrt{n}\ |\bar{x}|,\ \frac{w}{2}\sqrt{\frac{n}{n-1}}\right).$$

The following result is due to Laradji and Joarder (2002).

**Theorem 2.4** For any sample of $n \ge 2$ observations $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, the following inequalities hold:

$$(i)\ \frac{1}{2}\left[\left(1+\frac{1}{n}\right)\tilde{x}+\left(1-\frac{1}{n}\right)x_{(1)}\right]\le\bar{x}\le\frac{1}{2}\left[\left(1+\frac{1}{n}\right)\tilde{x}+\left(1-\frac{1}{n}\right)x_{(n)}\right],$$

$$(ii)\ |\tilde{x}-\bar{x}|\ \le\ \frac{n-1}{n+1}\ \max\ (\bar{x}-x_{(1)},x_{(n)}-\bar{x}),$$

$$(iii)\ \left|\frac{\tilde{x}}{\bar{x}}-1\right|\le 1.$$

## 3. Inequalities for some Useful Statistics

The following corollaries are obvious from Theorem 2.2 and Corollary 2.4 respectively.

**Corollary 3.1** If $n\ge 2$ observations are positive, then the coefficient of variation $CV(x)=s/\bar{x}$ satisfies the following inequalities:

$$\left|\frac{\tilde{x}}{\bar{x}}-1\right|\le\sqrt{\frac{n}{n-1}}\left|\frac{\tilde{x}}{\bar{x}}-1\right|\le CV(x)\le\sqrt{n}\ .$$

**Corollary 3.2** If $n\ge 2$ observations are positive, then the coefficient of skewness $CS(x)=\dfrac{\bar{x}-\tilde{x}}{s/3}$ satisfies the following inequalities:

$$-3\sqrt{\frac{n-1}{n}}\le CS(x)\le 3\sqrt{\frac{n-1}{n}}$$

which is slightly narrower than the known interval $[-3,3]$.

**Theorem 3.1.** Let $w_y=y_{(n)}-y_{(1),}w_x=x_{(n)}-x_{(1)},s_{xy}=\sum(x-\bar{x})(y-\bar{y}),s_{xx}=s_x^2$**.** Then the regression coefficient $\hat{\beta}_1=s_{xy}/s_{xx}$ satisfies the following inequalities:

$$(i)\ -\frac{s_y}{s_x}\le\hat{\beta}_1\le\frac{s_y}{s_x}\ ,$$

$$(ii)\ -\sqrt{\frac{n}{2}}\ \frac{w_y}{w_x}\le\hat{\beta}_1\le\sqrt{\frac{n}{2}}\ \frac{w_y}{w_x}\ .$$

**Proof.** The sample correlation coefficient $(r)$ is defined by $rs_xs_y=s_{xy}=\hat{\beta}_1s_{xx}$ so that the inequality in (i) follows by virtue of $-1\le r\le 1$. The proof for part (ii) is immediate by virtue of Theorem 1.2.

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**Theorem 2.1** Let $\bar{x}, \tilde{x}, w$ and $s$ respectively denote the mean, median, range and standard deviation of a sample of size $n$. Then

$$\frac{E\left(w^2\right)}{2(n-1)} \le \frac{E\left(w^2\right)}{2(n-1)} + \frac{E(\tilde{x}-\bar{x})^2}{2} \le E\left(s^2\right) \le \frac{nE(\bar{x}-x_{(1)})(x_{(n)}-\bar{x})}{n-1} \le \quad \frac{E\left(w^2\right)}{4} \quad \frac{n}{n-1}.$$

Since $E\left(s^2\right)=\sigma^2$, and hence the above inequality is a fixed interval for any population.