Chapter 4
Mathematical Expectation

4.1 Mean of a Random Variable

Objectives:

1. **To define the mean of a random variable.**
2. **To define the mean of a function of a random variable.**

Definition: Let X be a r.v. with a pdf $f(x)$. The **mean** or the **expected value** or the **average** of X is defined as:

a. $\mu = E(X) = \sum_{\forall x} x f(x)$, if X is discrete.

b. $\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$, if X is continuous.

Ex.1 (4.1/89): If 3 components to be drawn without replacement from a box of 4 good and 3 defective components, then find the expected value of the number of good components. Let X: the number of good components, then $X = \{0, 1, 2, 3\}$.

Comment: If a sample of size 3 is selected from that lot a big number of times, and then on the average it would contain 1.7 good items.

Ex.2 (4.2/90): A fair coin is tossed 3 times. The gambler wins \$5 if he gets all the same and loses \$3 otherwise. Find the expected amount the gambler can win.

Ex.3 (4.3/91): Find the expected life of this type of device.

Theorem: Let X be a r.v. with pdf $f(x)$ then the mean or expected value of the r.v. $g(X)$ is:

a. $\mu_{g(X)} = E(g(X)) = \sum_{\forall x} g(x) f(x)$, if X is discrete.

b. $\mu_{g(X)} = E(g(X)) = \int_{-\infty}^{+\infty} g(x) f(x) dx$, if X is continuous.

Ex.4: Problem (15/94). Find $E(X)$ & $E(X^2)$ where $f(x) = \begin{cases} x \ , & 0 < x < 1 \\ 2 - x \ , & 1 \le x < 2 \\ 0 \ , & elsewhere \end{cases}$

Ex.5: Problem (17/95).

X	– 3	6	9
***f*(*x*)**	1/6	1/2	1/3

Find $E(g(X))$ where $g(x) = (2x+1)^2$

Ex.6: Problem (18/95). Refer to problem (2/94) and find $E(X^2)$ where

X	0	1	2	3
***f*(*x*)**	27/64	27/64	9/64	1/64

Ex.7: Problem (20/95). Find $E(e^{2X/3})$ where $f(x) = \begin{cases} e^{-x} \ , & 0 < x \\ 0 \ , & elsewhere \end{cases}$.

4.2 Variance of a Random Variable

Objectives:

1. **To define the variance of a random variable.**
2. **To define the variance of a function of a random variable.**

Definition: Let X be a r.v. with a pdf $f(x)$ and mean μ. The **variance** of X, denoted by σ^2 is defined as;

a. $V(X) = \sigma^2 = E[(X-\mu)^2] = \sum_{\forall x} (x-\mu)^2 f(x)$, if X is discrete.

b. $V(X) = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx$, if X is continuous.

Theorem: The variance of X can be computed as;

$$\sigma^2 = E(X^2) - (E(X))^2 = E(X^2) - E^2(X) = E(X^2) - \mu^2$$

Ex.1: Problem (2/102). Find the standard deviation of X where;

X	– 2	3	5
***f*(*x*)**	0.3	0.2	0.5

Ex.2: Problem (3/102). Find the variance of X where;

x	2	3	4	5	6
***f*(*x*)**	0.01	0.25	0.4	0.3	0.04

Ex.3: Refer to problem (15/94). Find μ and σ^2.

Definition: Let X be a r.v. with a pdf $f(x)$ and mean μ. The **variance** of the r.v. $g(X)$, denoted by $\sigma^2_{g(X)}$, is defined as;

a. $V[(g(X)] = \sigma^2_{g(X)} = E[(g(X) - \mu)^2] = \sum_{\forall x} \left(g(x) - \mu_{g(X)}\right)^2 f(x)$, if X is discrete.

b. $V[(g(X)] = \sigma^2_{g(X)} = E(g(X) - \mu)^2 = \int_{-\infty}^{+\infty} \left(g(x) - \mu_{g(X)}\right)^2 f(x)dx$, if X is continuous.

Ex.4: Refer to Ex.1 and find the variance of $g(X) = 2X + 3$.

Ex.5: Refer to Ex.3 and find the variance of $Y = 3X + 2$.

4.3 Means of Linear Combinations of Random Variables

Objectives:

To find the mean of a linear combination of random variables.

Theorem: If a & b are constants, then $E(aX+b) = aE(X) + b$.

*Corollary*1: If $a = 0$, then $E(b) = b$.

*Corollary*2: If $b = 0$, then $E(aX) = aE(X)$.

Ex.1: Problem (2/102). Find the mean of $g(X) = 2X + 3$.

Theorem: If a & b are constants and $g(X)$ & $h(X)$ are functions of the r.v. X, then $E[ag(X) \pm bh(X)] = aE[g(X)] \pm bE[h(X)]$.

Ex.2: Refer to problem (15/94). Find the mean of $Y = (X - 1)^2$.

Theorem: If a & b are constants, then $V(aX+b) = a^2V(X)$.

*Corollary*1: If $a = 0$, then $V(b) = 0$.

*Corollary*2: If $b = 0$, then $V(aX) = a^2V(X)$.

Ex.3: Problem (7/113). If a r.v. X is defined such that $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$, then find μ and σ^2.