Chapter 3
Random Variables and Probability Distributions

3.1 Concept of a Random Variable
Objectives:

1. **To define the random variable (r.v.).**
2. **To determine the type of the random variable.**

Definition: The **random variable** (r.v.) is a function that associates a real number with each element in the sample space. It is usually denoted by T, U, V, W, X, Y, or Z and its value is denoted by the lower case letters. Simply, $X : \boldsymbol{S} \to \boldsymbol{R}$.

Types of Random Varibales

a. **Discrete r.v.**: It takes on a set of countable, finite or infinite, set of outcomes, and so its values are countable.

b. **Continuous r.v.**: It takes on values on a continuous scale, as intervals, uncountable sets of real numbers.

Note: The sample space can be also discrete or continuous.

Ex.1 (3.1/64): Y: the number of red balls ➔ Y = 0, 1, 2.

Note: In most practical problems

a. **Discrete r.v.** represent measured data such as height, weight, and temperature.

b. **Continuous r.v.** represent countable data, such as the number of heads in a coin tossing or the number of elements of anything.

Ex.2: Problem (2/72). Let the r.v. X: # of automobiles with paint blemishes if B: Automobile with blemish.

3.2 Discrete Probability Distributions
Objectives:

1. To define the discrete probability distribution function (pdf) and the cumulative distribution function (cdf).
2. To sketch the graph of the discrete probability distribution and the cumulative distribution.

Definition: The **probability distribution or function** of a discrete r.v. is the set of ordered pairs (x, $p(X=x)=f(x)$), and $f(x)$ is called the **probability distribution function (pdf)** and it satisfies;

1. $0 < f(x) < 1$. 2. $\sum_{\forall x} f(x) = 1$. 3. $P(X=x)=f(x)$.

Note: The pdf can be written as a table of $f(x)$ ands its corresponding x;

x	x_1	x_2	. . .	x_k	Total
$f(x)$	$f(x_1)$	$f(x_2)$	. . .	$f(x_k)$	1

Ex.1 (3.3/66): The box has 3 D and 5 N. Let X: 3 of D in a sample of size 2 and find the pdf of X.

Ex.2 (3.4/66): Let Y: # of cars with airbags, and P(cars with airbags) = 0.5. Find the pdf of Y.

Definition: The **cumulative distribution function (cdf)**, denoted by $\boldsymbol{F(x)}$, of a discrete r.v. with a pdf $f(x)$ is defined as;

$$F(x) = P(X \le x) = \sum_{\forall t \le x} f(t), \quad -\infty < x < \infty$$

Ex.3 (3.5/67): Refer to Ex.2 and find the cdf of *Y*. Using $F(y)$ verify that $f(2) = 3/8$.

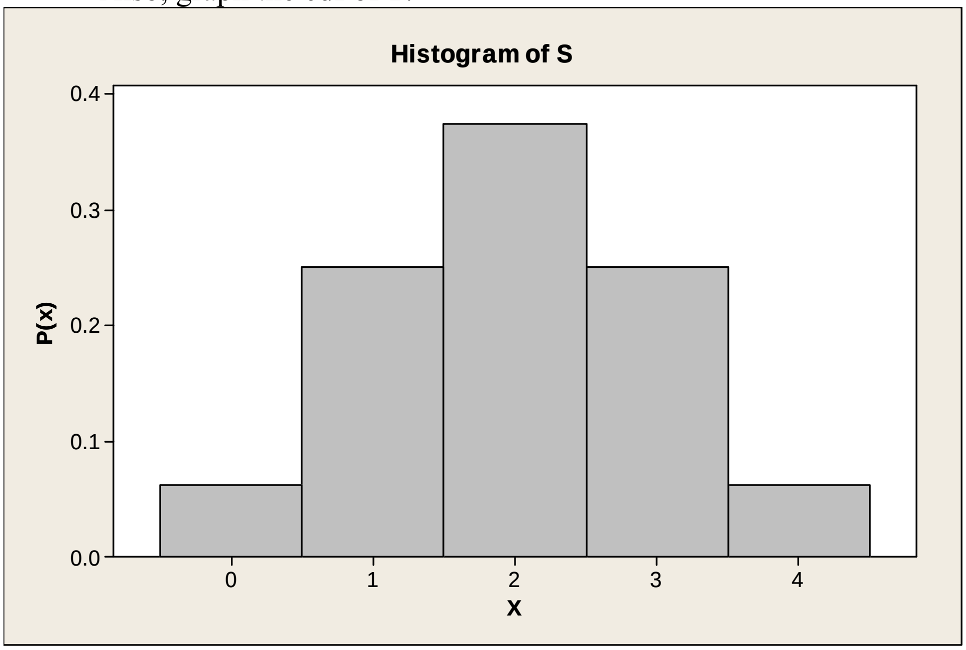
y	0	1	2	3	4	**Total**
$f(y)$	1/16	4/16	6/16	4/16	1/16	**1**
$F(y)$	1/16	5/16	11/16	15/16	16/16	

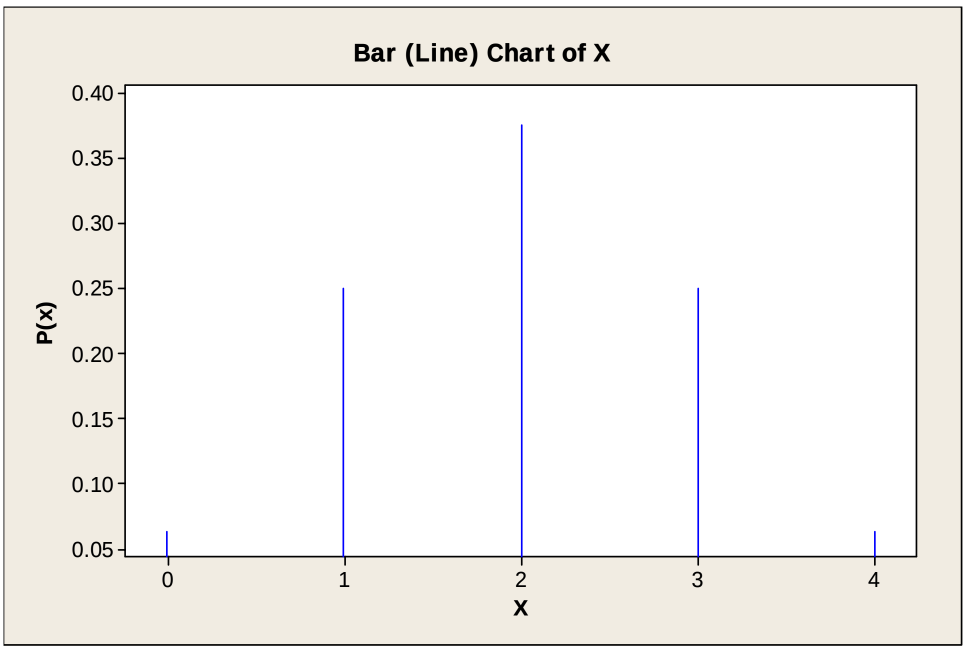
Note: $f(x) = F(x) - F(x\text{-}1)$.

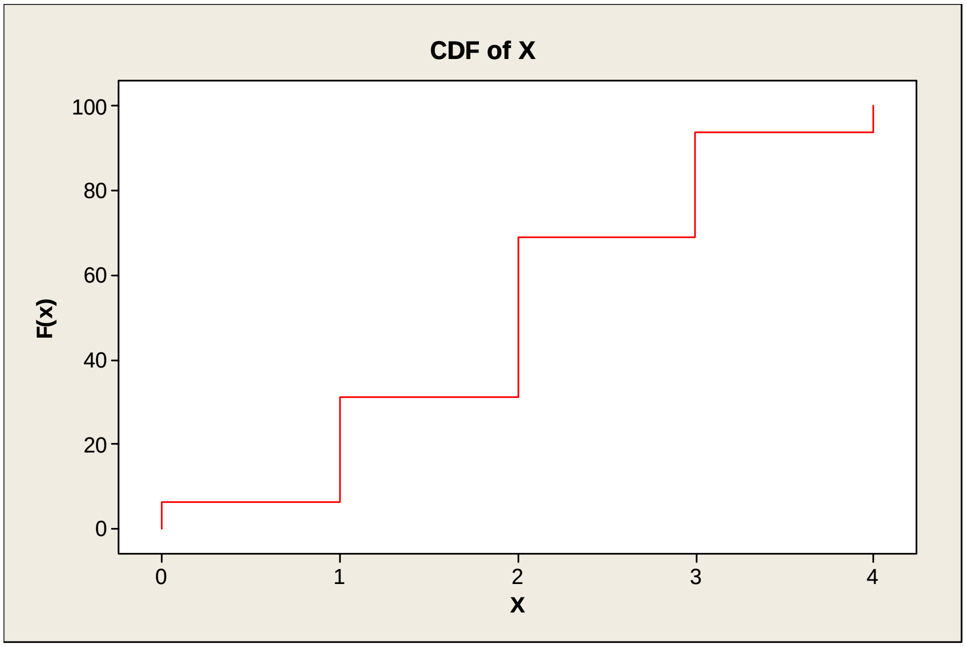
Graphs of a pdf and a cdf

The pdf can be described by a 1. Bar (Line) Chart   2. Prob. Histogram, where the cdf is described as a step function.

Ex.4: Refer to Ex.3 and graph the pdf of *Y* by a bar chart and a histogram. Also, graph the cdf of *Y*.







Ex.5: Given that the pdf of a r.v. X is given by

$f(x) = c(x^2 - 1)$, for $x = 2,3,4$, then find the value of c.

Ex.6: Problem (10/73). Find the pdf for the outcome of rolling a die once.

Ex.7: Problem (12/73). $F(t) = \begin{cases} 0 & , \quad t < 1 \\ \frac{1}{4} & , \ 1 \le t < 3 \\ \frac{1}{2} & , 3 \le t < 5 \\ \frac{3}{4} & , \ 5 \le t < 7 \\ 1 & , \quad t \ge 7 \end{cases}$, find $f(t)$.

3.3 Continuous Probability Distributions

Objectives:

1. To define the continuous probability density function (pdf) and the cumulative distribution function (cdf).

2. To calculate the probabilities of a continuous r.v. using the cdf.

Definition: The **probability density function** of a continuous r.v. X is denoted by $f(x)$ and is defined over the set of real numbers $\mathcal{R}$ if it satisfies;

1. $0 < f(x), \forall x \in \Re$. 2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

3. $P(a < X < b) = \int_a^b f(x)dx$.

Note:

a. $P(a < X \le b) = P(a \le X < b) = P(a \le X \le b) = P(a < X < b) = \int_a^b f(x)dx$

b. $P(X = a) = \int_a^a f(x)dx = 0$, for any real number a.

c. A continuous r.v. can NOT be presented in tabular form. It is presented as a formula.

Ex.1: Let X be a r.v. having the pdf $f(x)=\begin{cases} kx & , \quad 0 \le x \le 2 \\ 0 & , \quad elsewhere \end{cases}$, then find

a. The value of k. b. $P(1 \le X \le 1.5)$ c. $P(x > 0.5)$

d. $P(X < 4)$ e. $P(A)$ where $A = \{0, 1, 1.5, 2\}$

Definition: The cumulative distribution function (cdf), denoted by $F(x)$, of a continuous r.v. X with a pdf $f(x)$ is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt \ , \ -\infty < x < \infty .$$

Note:1. $f(x) = \frac{d}{dx} F(x) = F'(x)$

2. $P(a < X < b) = \int_{a}^{b} f(x)dx = F(b) - F(a)$

Ex.2: Let $f(x)=\begin{cases} 2e^{-2x} & , \quad x \ge 0 \\ 0 & , \quad elsewhere \end{cases}$ then find $F(x)$ and $P(1 \le X < 5)$.

Ex.3: Let $f(x)=\begin{cases} x & , \ 0 \le x \le 1 \\ 2-x & , \ 1 \le x \le 2 \end{cases}$ then find $F(x)$ and $P(0.5 \le X \le 1.5)$.

Ex.4: Problem (14/73).