Chapter Five

Discrete and Continuous Probability Distributions

Chapter Goals:

After completing this chapter, you should be able to

- Apply the binomial distribution to applied problems.
- Compute probabilities for The Poisson and Hypergeometric distributions.
- Find the probabilities using a normal distribution table and the normal business problems.
- Recognize when to apply the uniform and exponential distributions.

Probability Distributions:

1. Discrete Probability Distributions:

1. Binomial Distribution.
2. Poisson Distribution.
3. Hypergeometric Distribution.

Discrete Random Variable: it is random variables that can assume only countable number of possible values.

Examples: 1. Number of students who late to the class.
2. Number of T.V in a household.
3. Number of customers who arrive at a store.

In general: the outcome only TWO possible outcome.

Gender: Male or Female

Defective: Yes or No

2. Continuous Probability Distributions:

1. Normal Distribution.
2. Uniform Distribution.
3. Exponential Distribution.

Continuous Random Variable: it is random variables that can assume any value in interval. (**Uncountable number of possible values**)

Examples: 1. height and length
2. time required to complete a job
3. temperature.

In general: These can potentially take on any value, depend only on the ability to measure accurately.

5.1: *The Binomial Probability Distribution:*

Characteristics of the Binomial Distribution:

1. A trial has only two possible outcomes "Success" or "Failure"
2. There is fixed number of trials "n".
3. The trials of the experiment are independent of each other.
4. The probability of "Success" for each trial is p, and the probability of "Failure" is q = 1 - p

Binomial Distribution Formula:

$$P(x) = C_x^n p^x (1-p)^{n-x} \quad , \quad x = 0,1,2,\ldots n$$

Where:

$C_x^n = \dfrac{n!}{x!(n-x)!}$ Number of combinations of x objectives from n objectives.

$n! = n(n-1)(n-2)\ldots(2)(1)$, $x! = x(x-1)(x-2)\ldots(2)(1)$, $0! = 0$

and

$P(x)$: Probability of x successes in n trials.

x: Number of successes in the experiment.

n: Number if trials (sample size).

p: Probability of successes for each trial.

Mean and Variance for the Binomial Distribution:

Mean: $\mu = np$

Variance: $\sigma^2 = np(1-p)$ and *standard deviation* $\sigma = \sqrt{np(1-p)}$

Example: Tossing a coin 5 times, Let X number of H comes up,

a. write the sample space

b. Find probability to get 3 H exact.

c. Find the probability at most 3 H exist.

d. Write the probability distribution for the random variable x.

e. Find the mean and the standard deviation.

Example: An Urn contain 10 calculators, 4 of the calculators are defective. If sample of size 4 randomly selected, Find

a. The probability that 2 of them defective

b. The probability that at least 2 defective.

c. The mean and the standard deviation.

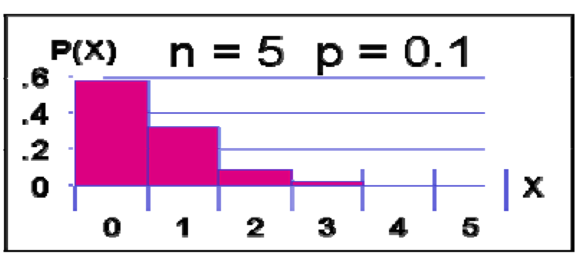
Example An oil drilling company ventures into various locations, and their success or failure is independent from one location to another. Suppose the probability of a success at any specific location is 0.1. If a driller drills 5 locations,

a. find the probability that there will be at least two successes locations.

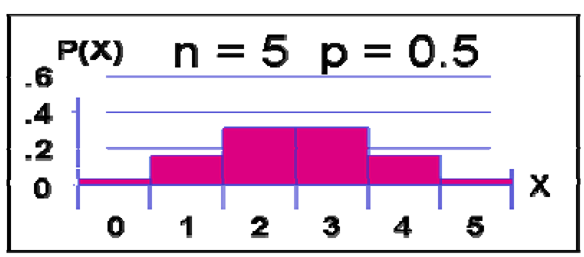
b. Find the probability that between 2 and 4, inclusive, successes locations.

Note:

The shape of the binomial distribution depend on the values of n and p. if p = 0.5 then the shape is bell – shaped regardless the size of the sample



Here n = 5 and p = 0.1



Here n = 5 and p = 0.5

5.2: ***The Poisson Probability Distribution:***

Characteristics of the Poisson Distribution:

1. The outcomes of interest are relative to the possible outcomes.
2. The average number of outcomes of interest per time or space interval is λ.
3. The number of outcomes of interest are random, and the occurrence does not influence the chance of another outcome of interest.
4. The probability of that an outcome of interest occurs in a given segment is the same for all segments.

Poisson Distribution Formula:

$$P(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \ldots$$

Where:

x: Number of successes in segment of interest.

t: Size of the segment of interest.

λ: Expected number of successes in a segment of unit size.

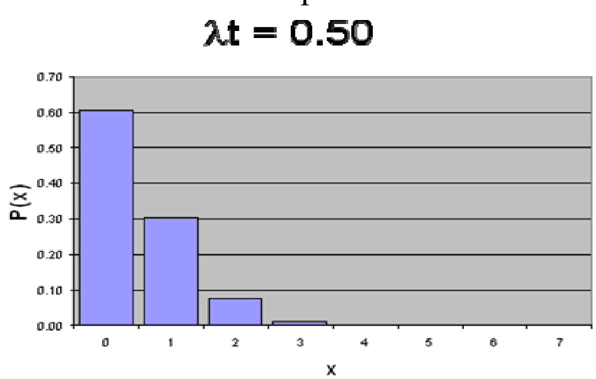
e: Base of the natural logarithm system .

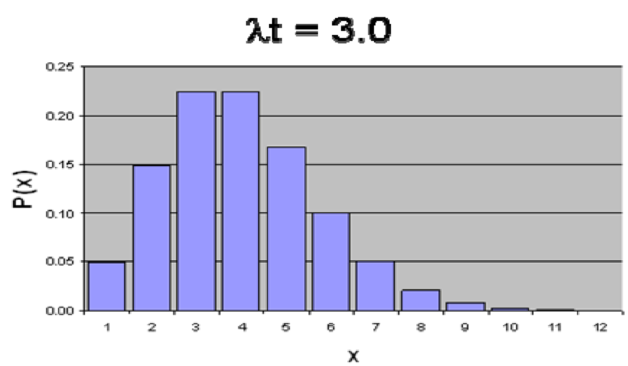
Mean and Variance for the Binomial Distribution:

Mean: $\mu = \lambda t$

Variance: $\sigma^2 = \lambda t$ and standard deviation $\sigma = \sqrt{\lambda t}$

Note: The shape of the Poisson distribution depends on the parameters λ and t:





Example: At a checkout center customers arrive at an average of 2.5 per minute.

a. What is the probability that 2 will arrive within 2 minutes?

b. What is the probability that at least 2 will arrive within 2 minutes?

c. Find the standard deviation of the number customers who will arrive to the desk within one and half hour.

Example: The number of failures of a testing instrument from contamination particles on the product is a Poisson random variable with a mean of 0.02 failures per hour.

a. What is the probability that the instrument does not fail in an 8-hour shift?

b. What is the probability of at least one failure in 30 minutes?

Example: On average, a certain intersection results in 3 traffic accidents per month. What is the probability that for any given month at this intersection

a. exactly 5 accidents will occur in a months?

b. less than 3 accidents will occur in two months?

c. 2 or 3 accidents will occur on three nomths?

d. Find the median of the traffic accidents per month.

The Hypergeometric Distribution

- "n" trials in a sample taken from a finite population of size N
- Sample taken without replacement
- Trials are dependent
- Concerned with finding the probability of "x" successes in the sample where there are "X" successes in the population

Hypergeometric Distribution Formula

$$P(x) = \frac{C_{n-x}^{N-X} \; C_{x}^{X}}{C_{n}^{N}}$$ (Two possible outcomes per trial)

Where

N = Population size
X = number of successes in the population
n = sample size
x = number of successes in the sample
n – x = number of failures in the sample

Example: 3 Light bulbs were selected from 10. Of the 10 there were 4 defective. What is the probability that 2 of the 3 selected are defective?

Example: An Urn contain 10 calculators, 4 of the calculators are defective. If sample of size 4 randomly selected, without repacment, Find

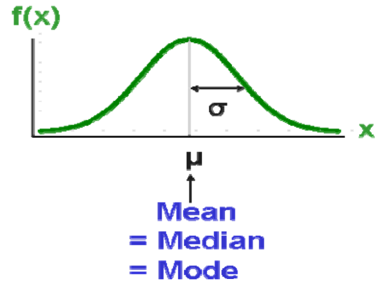
a. The probability that 2 of them defective

b. The probability that at least 2 defective.

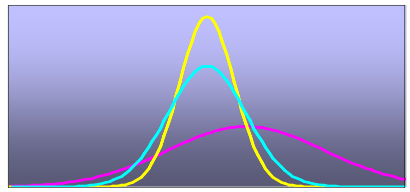
c. The probability that 2 or 3 defective.

5.3: ***The Normal Distribution:***

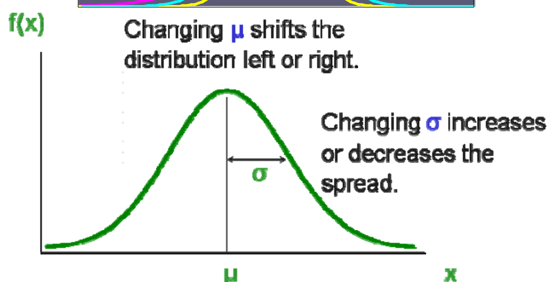
- Bell Shaped'
- Symmetrical
- Mean, Median and Mode are equal
- Location is determined by the mean, μ
- Spread is determined by the standard deviation, σ
- The random variable has an infinite theoretical range: $-\infty, +\infty$



Note: By varying the parameters μ and σ, we obtain different normal distributions

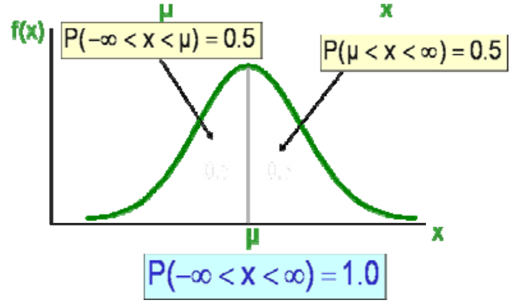


The Normal Distribution Shape



Finding Normal Probabilities

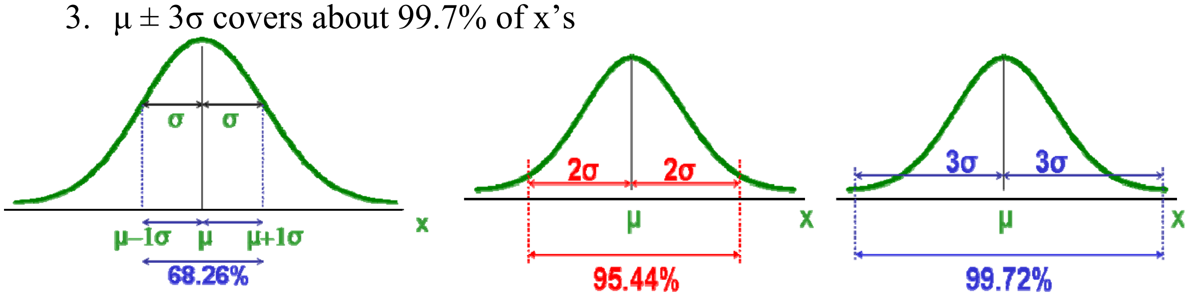
- Probability is measured by the area under the curve.
- The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



Empirical Rules

What can we say about the distribution of values around the mean? There are some general rules:

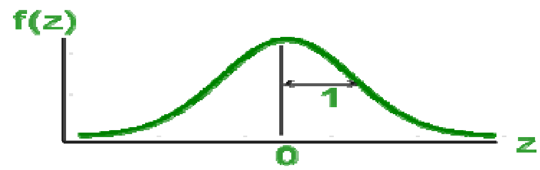
1. $\mu \pm 1\sigma$ encloses about 68% of x's
2. $\mu \pm 2\sigma$ covers about 95% of x's
3. $\mu \pm 3\sigma$ covers about 99.7% of x's



Importance of the Rule

- If a value is about 2 or more standard deviations away from the mean in a normal distribution, then it is far from the mean.
- The chance that a value that far or farther away from the mean is highly unlikely, given that particular mean and standard deviation

The Standard Normal Distribution



- Also known as the "z" distribution
- Mean is defined to be 0
- Standard Deviation is 1

Note:
Values above the mean have positive z-values,
values below the mean have negative z-values

Example: Let Z has a standard Normal distribution, Find the following:

a. $P(0 < Z < 1.34) =$

b. $P(Z > 2.56) =$

c. $P(Z < 1.78) =$

d. $P(1.45 < Z < 3.04) =$

e. $P(-0.93 < Z < 0) =$

f. $P(Z < -1.33) =$

g. $P(Z > -3.05) =$

h. $P(-2.45 < Z < -3.05) =$

i. $P(-1.45 < Z < 3.04) =$

Translation to the Standard Normal Distribution

- Any normal distribution (with any mean and standard deviation combination) can be transformed into the standard normal distribution (z)
- Need to transform x units into z units
- Translate from x to the standard normal (the "z" distribution) by subtracting the mean of x and dividing by its standard deviation:

$$z = \frac{x - \mu}{\sigma}$$

Example: If X is distributed normally with mean of 100 and standard deviation 50, find the following:

a. $P(X > 102) =$

b. $P(X > 96) =$

c. $P(85 < X < 104) =$

d. the median:

e. 80 percentile.

f. Find a such that $P(X > a) = 0.05$

g. If a sample of size 10 randomly selected, find the probability that 3 has a value at least 102.

5.4: ***The Uniform Distribution:***

The uniform distribution is a probability distribution that has equal probabilities for all possible outcomes of the random variable

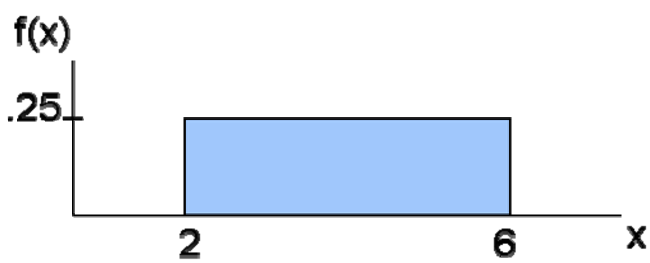
The Continuous Uniform Distribution

$$f(x)=\begin{cases}\dfrac{1}{\text{b}-\text{a}} & \text{if a} \le \text{x} \le \text{b}\\ 0 & \text{otherwise}\end{cases}$$

where

f(x) = value of the density function at any x value
a = lower limit of the interval
b = upper limit of the interval

Example: Uniform Probability Distribution over the range $2 \le x \le 6$:



The mean and the standard deviation for the Uniform distribution:

Expected value (mean): $E(x)=\mu=\dfrac{a+b}{2}$

Standard deviation: $\sigma=\sqrt{\dfrac{(b-a)^2}{12}}$

Example: **Q5.69 Page 216:**

The Exponential Distribution:

Used to measure the time that elapses between two occurrences of an event (the time between arrivals)

Examples:

- Time between trucks arriving at an unloading dock
- Time between transactions at an ATM Machine
- Time between phone calls to the main operator

The Exponential Distribution:

A continuous R.V. that is exponential distribution has the probability density function given by:

$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$, where λ the mean time between events

Where 1. mean $\mu = \dfrac{1}{\lambda}$

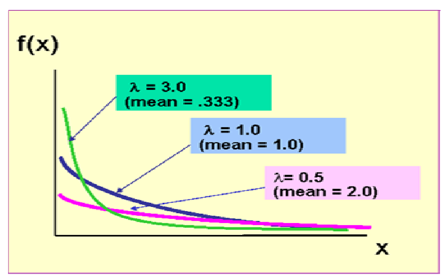
2. variance $\sigma^2 = \dfrac{1}{\lambda^2}$, and the standard deviation $\sigma = \dfrac{1}{\lambda}$

Notes:

1. The probability that an arrival time is equal to or less than some specified time a is $P(0 \le x \le a) = 1 - e^{-\lambda a}$
2. The probability that an arrival time between two specified time a is $P(b \le x \le a) = e^{-\lambda b} - e^{-\lambda a}$
3. The probability that an arrival time is grater than some specified time a is $P(x > a) = e^{-\lambda a}$
4. Note that if the number of occurrences per time period is Poisson with mean λ, then the time between occurrences is exponential with mean time $1/\lambda$

Shape of the exponential distribution

For any exponential destribution, with density function $f(x)$, $f(0) = \lambda$, and as x increase, $f(x)$ approaches to zero



Example: Life length of a particular type of battery follows exponential distribution with mean 200 hours. Find the probability that the

a. life length of a particular battery of this type is less than 200 hours.

b. life length of a particular battery of this type is more than 400 hours, given that it lives more that 300 hourd.

c. life length of a particular battery of this type is less than 200 hours or more than 400 hours.

Example: Suppose that on the average there are 5 car accidents weekly at the 3rd street. Then:

a. Find the probability that there will be at most 1 car accidents at the 3rd street next week.

b. Find the probability that there will be at least 2 car accident at the 3rd street in the coming 2 weeks.

c. Find the mean and standard deviation of the number of car accidents in two years.(Hint: Use one year =53 weeks)