Chapter 4
Using Probability and Probability Distributions

Chapter Goals:

After completing this chapter, you should be able to:

- To explain three approaches to assessing probabilities.
- To apply common rules of probability.
- To use Bayes' Theorem for conditional probabilities.
- To distinguish between discrete and continuous probability distributions.
- To compute the expected value and standard deviation for a discrete probability distribution.

4.1 The Basics of Probability:

Important Terms

- *Probability*: The chance that an *uncertain* event will occur (between 0 and 1).
- *Random Experiment*: A process of obtaining outcomes for uncertain events. Any experiment in which the coming outcome, although expected, cannot be determined in advance.
- *Sample Space*: The collection (set) of all possible elementary outcomes.
- *Elementary Event*: The most basic outcome possible from a random experiment. A single element from a sample space.

Sample Space:

Example1: Tossing a coin one time, write the sample space

$S = \{H,T\}$.

Example2: Tossing a coin two times, write the sample space

$S = \{(H,H),(H,T),(T,H),(T,T)\}$.

Example3: Rolling a die one time, write the sample space

$S = \{1,2,3,4,5,6\}$.

Example 4: Rolling a die two times, write the sample space

$$S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$$

Compound Events:

It is collection of more than one elementary event, where the elementary events consist of only one outcome from the sample space.

Example: Write all the elementary events for ex.1 to ex.4 above.

Solutions:

For ex. 1: $e_1 = \{H\}, e_2 = \{T\}$.

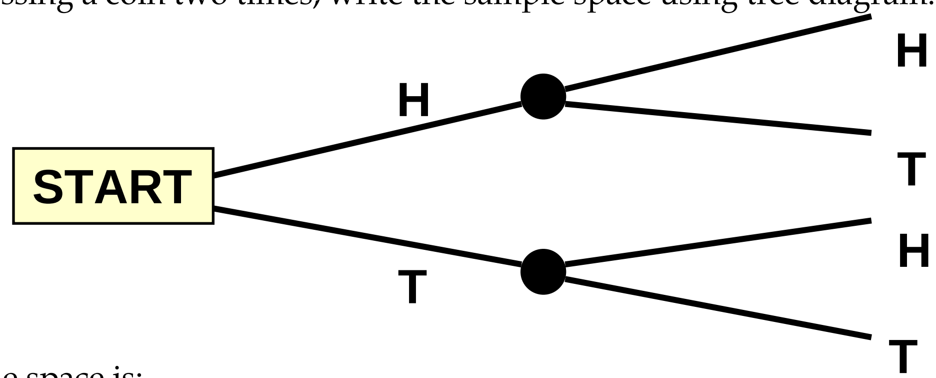
For ex. 2: $e_1 = \{HH\}, e_2 = \{HT\}, e_3 = \{TH\}, e_4 = \{TT\}$.

For ex. 3: $e_1 = \{1\}, e_2 = \{2\}, e_3 = \{3\}, e_4 = \{4\}, e_5 = \{5\}, e_6 = \{6\}$.

Writing the sample space using a tree diagram:

Example: Write the sample space for all examples from ex.1 to ex.4.

Example: Tossing a coin two times, write the sample space using tree diagram.



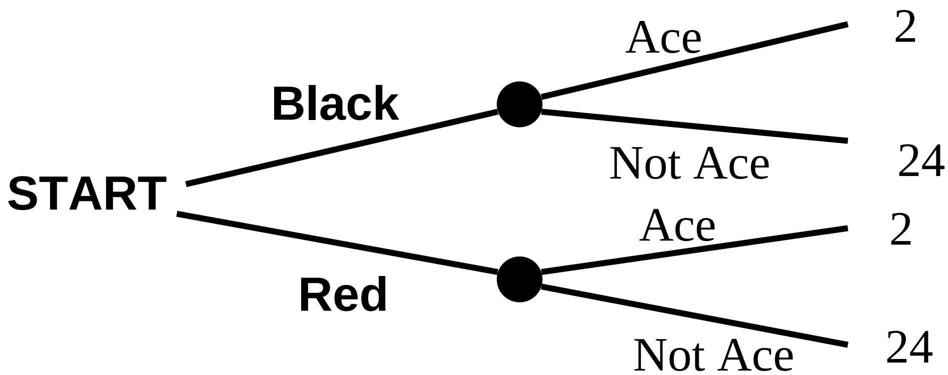
So the sample space is:

S = {(H,H),(H,T),(T,H),(T,T)}.

Writing the sample space using Contingency table (two-way table):

Example: Drawing a *red* card from a deck of cards and that card is an *ace*:

	Ace	Not Ace	Total
Black	2	24	26
Red	**2**	24	26
Total	4	48	52



Elementary Events

A automobile consultant records fuel type and vehicle type for a sample of vehicles:

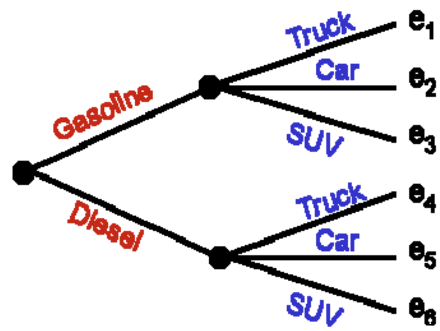
2 Fuel types: Gasoline, Diesel.

3 Vehicle types: Truck, Car, SUV.

6 possible elementary events:

e1	Gasoline, Truck
e2	Gasoline, Car
e3	Gasoline, SUV
e4	Diesel, Truck
e5	Diesel, Car
e6	Diesel, SUV

$S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.



Probability Concepts

1. ***Mutually Exclusive Events***

   The events are mutually exclusive if they cannot occur together.
   - If the event E_1 occurs, then the event E_2 cannot occur.
   - The events E_1 and E_2 have no common elements.

Example:
let the event *E1*: the card is *red* and *E2*: the card is *black* then E_1 and E_2 are M.E since its impossible to be red and black at the same time.

Example: Let $S = \{e_1, e_2, ..., e_8\}$ and $E_1 = \{e_1, e_3, e_5, e_7\}$, $E_2 = \{e_2, e_4, e_6, e_8\}$, are the two event Mutually exclusive? YES.
Solution:
Since E_1 and E_2 have no common elements then the two events are mutually exclusive.

2. ***Independent and Dependent Events***
   - *Independent*: The two events are independent if the occurrence of one *does not affect* the occurrence of the other, so it dose not affect the probability of the other.
   - *Dependent*: The two events are dependent if the occurrence of one affects the probability of the other.

Example for Independent Events:

E_1 = Getting a head on one flip of a fair coin.

E_2 = Getting a head on the second flip of the same coin.

Result of the second flip does not depend on the result of the first flip.

Example for Dependent Events:

E_1 = An *odd* number occurs when a die is rolled.

E_2 = An *even* number occurs when a die is rolled.

Probability of the second event is affected by the occurrence of the first event.

Assigning Probability

There are three methods to assign probability for the events.

1. *Classical Probability Assessment* (Exact Prob.):

$$P(E_i) = \frac{\text{Number of ways } E_i \text{ can occur}}{\text{Total number of elementary evens}}$$

2. *Relative Frequency of Occurrence* (Approx. Prob.):

$$P(E) \approx \text{Relative freq. of } E_i = \frac{\text{Number of times } E_i \text{ occur}}{N}$$

3. *Subjective Probability Assessment*

   It is a measure of personal convention that an outcome will occur. It is an opinion or judgment by a decision maker about the likelihood of an event.

Example: Q.4.1 Page 137:

a. P(Male) = # males / Total = 678/1,336 = 0.5075

b. P(20-40) = # 20-40/Total = 630/1,336 = 0.4716

c. P(20-40 and Male) = 340/1,336 = 0.2545

d. $P(<20 \mid Males) = \frac{\# < 20}{\# \, Males} = \frac{168}{678} = 0.2478$

$$P(<20 \mid Females) = \frac{\# < 20}{\# \, Females} = \frac{208}{658} = 0.3161$$

Gender and age are not independent

Example: Q.4.2 Page 137:

a. P(Brown) = # Brown/Total = 310/982 = 0.3157

b. P(YZ-99) = #YZ-99/Total = 375/982 = 0.3819

c. P(YZ-99 and Brown) = 205/982 = 0.2088

d. The two events are not mutually exclusive since their joint probability is 0.1324. It is possible to choose a White YZ-99 product. To be mutually exclusive the outcome for one event (e.g., White) means the other outcome (YZ-99) cannot occur.

Example: Q.4.6 Page 137:

The elementary events are listed below:

Elementary Event	Customer 1	Customer 2
Event 1	Like It	Like It
Event 2	Like It	Don't Like It
Event 3	Don't Like It	Like It
Event 4	Don't Like It	Don't Like It

Sample Space = {Event 1, Event 2, Event 3, Event 4}
Three events indicate a customer liking the product.

4.2. The Rules of Probability:

1. **Rules for Possible Values and Sum (Axioms of Probability):**
   a. Individual Values:
      $0 \le P(e_i) \le 1$, for any event e_i.
   b. Sum of All Values
      $$\sum_{i=1}^{k} P(e_i) = 1$$
   where:
      k = Number of elementary events in the sample space.
      e_i = i^{th} elementary event.

2. **Addition Rule for Elementary Events**
   If: $E_i = \{e_1, e_2, e_3, \ldots, e_k\}$
   then: $P(E_i) = P(e_1) + P(e_2) + P(e_3) + \ldots + P(e_k)$

3. **Complement Rule**
   $\bar{E}$ is the collection of all possible elementary events **not** contained in E, and its read "The complement of event E".
   $P(\bar{E}) = 1 - P(E)$ or $P(E) + P(\bar{E}) = 1$

4. **Addition Rule for Two Events**
   $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$
   Where:
   1. The word ***OR*** means the union $\bigcup$, all elements either in event E_1 or in event E_2 or both *without* repetition.
   2. The word ***AND*** means the intersection $\bigcap$, all elements that in both events at the same time.

5. **Addition Rule for Mutually Exclusive Events:**
   If E1 and E2 are mutually exclusive, then $P(E_1 \text{ and } E_2) = 0$
   $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$

6. **Conditional probability for any two events E_1 , E_2:**
   $$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad \text{where} \quad P(E_2) > 0$$

7. **Independent Events:**
   If E_1 and E_2 are two *independent* events then TFAE:
   $$P(E_1|E_2) = P(E_1) \text{ where } P(E_2) > 0$$
   $$P(E_2|E_1) = P(E_2) \text{ where } P(E_1) > 0$$
   Note: if $P(E_i \cap E_j) = P(E_i)P(E_j)$ then the two events are independent.

Note: If E_1, E_2 are independent, then 1. $\overline{E}_1, E_2$ 2. $E_1, \overline{E}_2$ 3. $\overline{E}_1, \overline{E}_2$ are independent.

Example: Let $S.S = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and $E_1 = \{e_1, e_3, e_5, e_7\}$, $E_2 = \{e_2, e_4, e_6, e_8\}$ $E_3 = \{e_1, e_3, e_6, e_8\}$, $E_4 = \{e_2, e_5, e_7\}$, and $P(e_1) = P(e_3) = 0.1$, $P(e_2) = P(e_4) = 0.15$ $P(e_5) = P(e_7) = 0.01$, $P(e_6) = 0.05$, then find

1. $P(e_8) =$
2. $P(E_1) =$
3. $P(E_3) =$
4. $P(\overline{E}_4) =$
5. $P(E_1 \text{ or } E_3) =$
6. $P(E_1 \text{ or } E_2) =$
7. $P(E_1 or E_3)^C =$
8. $P(E_1|E_4) =$
9. Are E_1 and E_4 independent? Explain.

Example: Q 4.16 page 156

a. P(A) = 1,000/2,100 = 0.4762

b. P(A and B) = 0 since A and B are mutually exclusive

c. P(B and F) = 300/2100 = 0.1429

d. $P(E|A) =$ 600/1000 = 0.60

e. P(A or F) = P(A) + P(F) - P(A and F)
   = 1000/2100 + 900/2100 - 300/2100
   = 0.4762 + 0.4286 - 0.1429
   = 0.7619

Example: Q 4.17 page 156

a. P(Dry) = P(Clear and Dry) + P(Cloudy and Dry)
   = 0.20 + 0.30
   = 0.50

b. P(Rainy or Cloudy & Dry) = P(Rainy) + P(Cloudy & Dry) – P(Rainy and Cloudy &Dry)

= 0.40 + 0.30 - 0.0 = 0.70

c. $P(Cloudy|Dry) = \frac{P(Cloudy\ and\ Dry)}{P(Dry)} = \frac{0.30}{0.50} = 0.60$

Example: Of the cars on a used car lot, 70% have air conditioning (AC), 40% have a CD player (CD), and 20% of the cars have both.

1. What is the probability that a car has a CD player, given that it has an AC ?

i.e., we want to find P(CD | AC)

Solution:

	CD	No CD	TOTAL
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
TOTAL	0.4	0.6	1.0

$$P(CD|AC) = \frac{P(CD\ and\ AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

2. If a car is selected randomly, what is the probability that the car has a CD?
3. If a car is selected randomly, what is the probability that the car has a AC?

Example: The following table represents number of students in KFUPM from Al-Sharqia or not from Al-Sharqia, and So or Jr

	Al-Sharqia	Not from Al-Sharqia
So	256	940
Jr	340	784

If a student randomly selected, find the following

1. The probability that the student is from Al-Sharqia.
2. The probability that the student is So.
3. The probability that the student is a So, given that he is from Al-Sharqia.

Example: Q 4.20 page 157

a. P(all 3 pick the same color) = P(3 Brown) + P(3 Gray) + P(3 Red)

P(3 Brown) = 1/27 (see part a.)

P(3 Grey) = 1/27

P(3 Red) = 1/27

P(all 3 same color) = 3/27 = 0.1111

Note: if each customer selects a color at random, the events are independent.

b. Use the compliment rule:

P(not all 3 will select the same color) = 1 – P(all 3 pick the same color).

P(not all 3 will select the same color) = 1 – 3/27 = 24/27 = 0.8889.

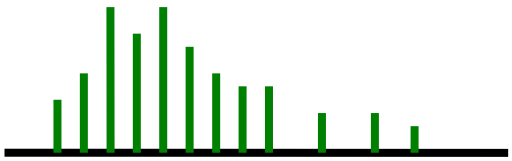
4.3. Introduction to Probability Distributions

Random Variable:

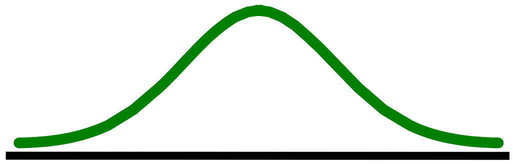
Represents a possible numerical value from a random experiment. Real-valued function defined on the sample space. That is assigning a real number to every elementary event in the sample space. And it is denoted by *X*, *Y*, *Z*.

Two classes of random variables:

Discrete random variable.



Continuous random variable.



Discrete Random Variables

A r.v. that can only assume a *countable* number of values.

Example: Rolling a fair die twice; let the r.v X: represent number of times 4 comes up. What are the possible values for X?

Solution:

$$S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$$

The values for $X = \{0, 1, 2\}$

Example: Toss a balanced coin 5 times. Let Y be the number of heads.

Then $Y = \{0, 1, 2, 3, 4, \text{or } 5\}$.

Discrete Probability Distribution

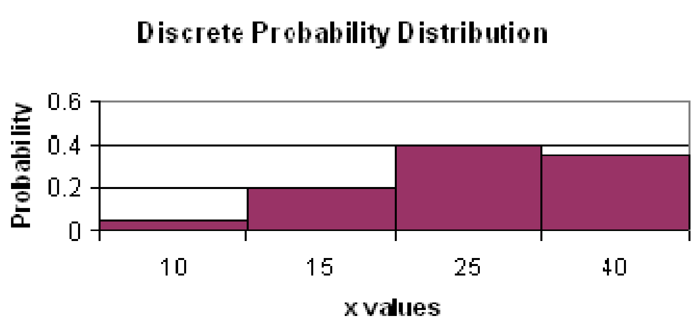
A list of all possible (xi , $P(x_i)$) pairs

x_i = Some value of Random Variable (Outcome).

$P(x_i)$ = Probability associated with this value.

Notes: the values of the r.v. represent a PARTITION of the sample space.

1. x_i's are mutually exclusive (no overlap).
2. x_i's are collectively exhaustive (nothing left out).
3. $0 \le P(x_i) \le 1 \quad for \quad all \quad x_i$
4. $\sum_{all\ i} P(x_i) = 1$



Discrete Random Variable Summary Measures

1. ***Expected Value of a discrete distribution*** ***(Weighted Average)***

$$\mu_X = E(X) = \sum_{All\ i} x_i P(x_i)$$

2. ***Standard Deviation of a discrete distribution***

$$\sigma_x = \sqrt{V(X)} = \sqrt{\sum \left[x - E(X)\right]^2 P(x_i)} = \sqrt{\sum_{All\ i} x_i^2 P(x_i) - \left[\sum_{All\ i} x_i P(x_i)\right]^2}$$

$$= \sqrt{\sum_{All\ x} x_i^2 P(x_i) - \left[E(X)\right]^2}$$

where:
$E(X)$ = Expected value of the random variable
x = Values of the random variable
$P(x)$ = Probability of the random variable having the value of x.

Example: Toss two *balanced* (*fair* or *unbiased*) coins, Let X = # of heads,

1. What are the possible values of the r.v. X.
2. Write down the probability distribution for the r.v. X.
3. Compute the expected value of X.
4. Compute the standard deviation of X.

Solution:

S = { HH,HT,TH,TT}

x	0	1	2	**Total**
$P(x_i)$	0.25	0.5	0.25	**1**
$x\,P(x)$	0	0.5	.5	**1**
$x^2\,P(x)$	0	0.5	1	**1.5**

The mean: $E(X) = \sum_{i=0}^{2} x_i P(x_i) = 0(0.25) + 1(0.5) + 2(0.25) = 1$

The standard deviation:

$$\sigma_x = \sqrt{V(X)} = \sqrt{\sum_{i=0}^{2} x_i^2 P(x_i) - \left[E(X)\right]^2} = \sqrt{0(0.25) + 1(0.5) + 4(0.25) - 1^2}$$

$$= \sqrt{0.5} = 0.707$$

Example: Q 4.40 Page 167

x	$P(x)$	$x\,P(x)$	$x^2\,P(x)$	$x-E(X)$	$[x-E(X)]^2$	$[x-E(X)]^2\,P(x)$
10	0.05	0.5	5	-17.5	306.25	15.3125
15	0.2	3	45	-12.5	156.25	31.25
25	0.4	10	250	-2.5	6.25	2.5
40	0.35	14	560	12.5	156.25	54.6875
		27.5	**860**			103.75

a. Mean = Expected value = 27.5
Variance = 103.75
Standard deviation = 10.1858

Example : Q 4.49 Page 168:
To determine the probability you need to find the relative frequency, that is to divide the number of days by the total of 200.

x	# of days	$P(x)$	$x\,P(x)$	$x-E(X)$	$[x-E(X)]^2$	$[x-E(X)]^2\,P(x)$
0	22	0.110	0.000	-2.885	8.3232	0.9156
1	20	0.100	0.100	-1.885	3.5532	0.3553
2	40	0.200	0.400	-0.885	0.7832	0.1566
3	55	0.275	0.825	0.115	0.0132	0.0036
4	28	0.140	0.560	1.115	1.2432	0.1741
5	20	0.100	0.500	2.115	4.4732	0.4473
6	5	0.025	0.150	3.115	9.7032	0.2426
7	10	0.050	0.350	4.115	16.9332	0.8467
TOTAL	**200**		**2.885**			**3.1418**

b. $E(X) = 2.885$
c. 1.7725
d. The coefficient of variation is:

$$CV = \frac{\sigma}{\mu}(100) = \frac{\sigma_X}{E(X)}(100) = \frac{1.7725}{2.885}(100) = 61.44\%$$

e. Look at the 3rd quartile. This is accomplished by determining the cumulative P(x). The 75th percentile would be between 3 and 4. Since you want at least 75% you should choose x= 4 which means you would need 4(3) = 12 employees.

x	$P(x)$	Cum $P(x)$
0	0.110	0.110
1	0.100	0.210
2	0.200	0.410
3	0.275	0.685
4	0.140	0.825
5	0.100	0.925
6	0.025	0.950
7	0.050	1.000