Chapter 10
One-and-Two-Sample Tests of Hypotheses

10.1 Statistical Hypotheses: General Concepts

Objectives:

1. **To introduce the statistical hypothesis.**
2. **To identify different types of statistical hypotheses.**

Definition: The **statistical hypothesis** is an assertion or conjecture concerning one or more populations.

Notes:

1. Usually we draw a random sample from the investigated population and use it to provide an evidence that either supports the hypothesis or does not.
2. Probability plays an important role in hypotheses testing since the acceptance of a hypothesis implies that the data do not give sufficient evidence to refute it, and rejection means to refute it. That is rejection means that there is a small probability of obtaining the sample information observed when, actually, the hypothesis is true.
3. There are two types of hypotheses; the **Null hypothesis** which we want to test **assuming it is true**, denoted by $\boldsymbol{H_0}$, and the **alternative hypothesis** which is against the null hypothesis, denoted by $\boldsymbol{H_1}$.
4. The null hypothesis has always an exact value of the population parameter, where the alternative hypotheses has several values, e.g. if H_0: $\mu = \mu_0$, then H_1: $\mu > \mu_0$, H_1: $\mu < \mu_0$, or H_1: $\mu \neq \mu_0$. consequently, the two hypotheses are always **disjoint** or exclusive.

10.2 Testing a Statistical Hypothesis

Objectives:

1. **To define the test statistic (function or ratio).**
2. **To define type-I and Type-II errors.**
3. **To compute type-I and Type-II errors.**

- Suppose that we want to test H_0: $p = ¼$ vs. H_1: $p > ¼$.
- The **test statistic** is a function based on the sample observations used to make a decision.
- Let $n = 20$, accept H_0 if $X \leq 8$, and reject H_0 if $X > 8$. So, $0 \leq X \leq 8$ is called an **acceptance region**, where H_0 if $20 \geq X > 8$ is called a **rejection region** or (**critical region**) and the value $X = 8$ is called a **critical value**.
- Depending on the sample data, rejecting or accepting the null hypothesis has, definitely, a certain margin of error. So, there are two types of errors in decision making as follows.
- **Type-I error** is rejecting the null hypothesis when it is, in fact, true.
- **Type-II error** is accepting H_0 when it is, in fact, false.

	Real status of H_0	
Decision	True	False
Reject H_0	**Type-I error**	**Correct**
Accept H_0	**Correct**	**Type-II error**

Definition: P(Type-I error) = α = Significance level = Test size.
Definition: P(Type-II error) = $\beta = P$(Accept H_0 | H_0 false).
Note: β cannot be computed unless we have a specific (simple or single-valued) alternative hypothesis.
Ex.1: For testing H_0: $p = ¼$ vs. H_1: $p > ¼$, compute α and β for specific value of H_1: $p > ¼$ say H_1: $p = ½$.
Note: α and β are balanced, i.e. increasing one of them will result in the reduction of the other. So, the only way to reduce both of them, reasonably, is by increasing the sample size.
Ex.2: Let $n = 100$, and H_0: $p = ¼$ vs. H_1: $p > ¼$. Find α and β if $X = 36$ is the critical value.
Note: Since α is more serious than β, in any statistical study α is always fixed and β is minimized with respect to α.
Ex.3: Compute α for testing H_0: $\mu = 68$ vs. H_1: $\mu \neq 68$, where the decision rule is: Reject H_0 if $\overline{X} < 67$ or $\overline{X} > 69$ and accept H_0 otherwise, assuming $n = 36$ and $\sigma = 3.6$. Compute β if H_1: $\mu = 70$.
Ex.4: Compute β if H_1: $\mu = 68.5$ ➔ $\beta = 0.8661$.
Note: Refer to the important properties of a test of a hypothesis in p. 293.
Definition: The **power** of the test is the probability of rejecting H_0 given that a specific alternative is true.
Note: Power = $1 - \beta$ and is used for comparing different types of tests.

10.3 One and Two-Tailed Tests

Objectives:

1. **To define the one and two-tailed tests.**
2. **To formulate the null and the alternative hypotheses.**

Definition:

1. The one-tailed (one-sided) test is defined as; H_0: $\theta = \theta_0$ vs. H_1: $\theta > \theta_0$ (Right-tailed test), or H_0: $\theta = \theta_0$ vs. H_1: $\theta < \theta_0$ (Left-tailed test).
2. The two-tailed (two-sided) test is defined as; H_0:θ=θ_0 vs. H_1:$\theta \neq \theta_0$.

Note:

1. For H_1: $\theta > \theta_0$ the critical region lies on the **right** tail of the distribution of the test statistic.
2. For H_1: $\theta < \theta_0$ the critical region lies on the **left** tail of the distribution of the test statistic.
3. For H_1: $\theta \neq \theta_0$ the critical region lies on **both** tails of the distribution of the test statistic with equal areas.

- The null hypothesis **always** has the format H_0: $\theta = \theta_0$.
- The alternative hypothesis H_1 is formulated depending on the problem statement. So it is, sometimes, called the researcher hypothesis.

Ex.1 (10.1/295): Let H_0: $\mu = 1.5$ vs. H_1: $\mu > 1.5$. Rejecting H_0 means that the test statistic value should be, enough, greater than 1.5 with a significance level α and accepting H_0 means that the data didn't provide a test statistic value that is, enough, greater than the hypothesized mean value.

Ex.2 (10.2/295): Let H_0: $p = 0.6$ vs. H_1: $p \neq 0.6$. This is a two-tailed test where the area of the critical region on each tail is $\alpha/2$.

10.4 The Use of *p*-values For Making a Decision

Objectives:

1. To define the *p*-value.

2. To compare between two testing approaches.

Definition: The *p*-value is the smallest level of significance at which the observed value of the test statistic is significant. Also, it is defined as the smallest level of significance at which the null hypothesis is rejected.

Note:

1. The *p*-value depends on the sample **size** and on the test **statistic**.
2. If α is not mentioned in the problem it is assumed **5%**, by default.

Comparison between two testing approaches

Classical testing approach	*P*-value approach
1. Formulate the **hypotheses**.	1. Formulate the **hypotheses**.
2. Find the **critical** region using α.	2. Calculate the **test statistic** value.
3. Calculate the **test** statistic value.	3. Compute ***p*-value** based on 2.
4. State the **decision rule**.	4. Compare 3 with α and **decide**.
5. Compare 2 with 3 & **decide**.	5. Draw your **conclusion**.
6. Draw your **conclusion**.	

10.5-10.7 Single Sample: Tests Concerning a Single Mean

Objectives:

1. To introduce the procedure for testing a single mean μ.

2. To consider different cases of testing a single mean μ.

3. To identify the relationship between C.I. & hypothesis testing.

Case.1: For a small sample, normal population, and known σ; the test statistic is: $Z_0 = \dfrac{\overline{X} - \mu_0}{SE(\overline{X})} = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$.

The Decision Rule:

First: Using the z-value;

1. (Right-Tailed) For H_1: $\mu > \mu_0$, Reject H_0 if $Z_0 > z_\alpha$.
2. (Left-Tailed) For H_1: $\mu < \mu_0$, Reject H_0 if $Z_0 < -z_\alpha$.
3. (Two-Tailed) For H_1: $\mu \neq \mu_0$, Reject H_0 if $|Z_0| > z_{\alpha/2}$.

Second: Using the p-value; Reject H_0 if p-value $< \alpha$, where;

1. (Right-Tailed) For H_1: $\mu > \mu_0$, p-value $= P(Z > Z_0)$.
2. (Left-Tailed) For H_1: $\mu < \mu_0$, p-value $= P(Z < Z_0)$.
3. (Two-Tailed) For H_1: $\mu \neq \mu_0$, p-value $= P(|Z| > |Z_0|)$.

Third: Using a C.I. only for two-tailed test; Reject H_0 if μ_0 is outside a $(1 - \alpha)$ 100% C.I. for μ.

Ex.1 (10.3/301): $n = 100$, $\overline{X} = 71.8$, $\sigma = 8.9$, $\alpha = 0.05$. Test H_0: $\mu = 70$ vs. H_1: $\mu > 70$ using different approaches.

Ex.2 (10.4/302): $n = 50$, $\overline{X} = 7.8$, $\sigma = 0.5$, $\alpha = 0.01$. Test H_0: $\mu = 8$ vs. H_1: $\mu \neq 8$ using different approaches.

Case.2: For a large sample, the same statistic for case 1 will be used, but s can simply replace σ when it is unknown.

Ex.3: Problem (3/319), $n = 64$, $\overline{X} = 38$, $s = 5.8$. Test H_0: $\mu = 40$ vs. H_1: $\mu < 40$, using different approaches.

Ex.4: Problem (5/319), $n = 100$, $\overline{X} = 23{,}500$, $s = 3{,}900$. Test H_0: $\mu = 20{,}000$ vs. H_1: $\mu > 20{,}000$, using different approaches.

Case.3: For a small sample, normal population, and unknown σ; the test statistic is: $T_0 = \frac{\overline{X} - \mu_0}{SE(\overline{X})} = \frac{\overline{X} - \mu_0}{s/\sqrt{n}}$. In this case, the normal distribution is replaced by the t distribution.

Ex.5 (10.5/304): $n = 12$, $\overline{X} = 42$, $s = 11.9$. Test H_0: $\mu = 46$ vs. H_1: $\mu < 46$, using different approaches.

Ex.6: Problem (8/319). $n = 20$, $\overline{X} = 244$, $s = 24.5$. Test H_0: $\mu = 220$ vs. H_1: $\mu > 220$ using different approaches.

10.11 Single Sample: Tests Concerning a Single Proportion

Objectives:

1. **To introduce the procedure for testing the population binomial parameter P for small and large sample sizes.**
2. **To consider different procedures for testing a single proportion.**

Let the r.v. X: the number of successes in n Bernoulli trials with a probability of success p.

Case.1: For a small sample size, X:B(n,p), and:

1. The hypotheses are; H_0: $p = p_0$ vs. H_1: $p >(< \text{ or } \neq)\ p_0$.
2. The test statistic is $p_0 = x/n \rightarrow x_0 = np_0$.
3. DR: Reject H_0 if p-value $\leq \alpha$, since the binomial r.v. is discrete.
4. Calculating the p-value depends on H_1, x, and x_0:
   a. If H_1: $p > p_0$ then p-value = $P(X \geq x \mid p_0)$.
   b. If H_1: $p < p_0$ then p-value = $P(X \leq x \mid p_0)$.
   c. If H_1: $p \neq p_0$ then p-value = $2P(X \geq x \mid p_0)$ for $x > x_0$.
   d. If H_1: $p \neq p_0$ then p-value = $2P(X \leq x \mid p_0)$ for $x < x_0$.

Ex.1 (10.10/325): $n = 15$, $x = 8$, and $\alpha = 0.1$.

Case.2: For a large sample size, X~N(np , npq) by the CLT, and:

1. The hypotheses are; H_0: $p = p_0$ vs. H_1: $p >(< \text{ or } \neq)\ p_0$.
2. The test statistic is $Z_0 = \frac{x - np_0}{\sqrt{np_0q_0}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0q_0}{n}}}$~N(0 , 1).

Ex.2 (10.11/325): $n = 100$, $x = 70$, and $\alpha = 0.05$.

Ex.3: Problem (1/328). $n = 20$, $x = 9$. One-tailed test.

Ex.4: Problem (6/328). $n = 90$, $x = 28$.