Chapter 2
Probability

2.1 Sample Space

Objectives:

1. **To define the sample space.**
2. **To describe the sample space using the tree diagram and the rule (set builder notation).**

Definition: The **random experiment** is an experiment in which the outcome can not be determined in advance although the possible outcomes have been pre-assigned. It is sometimes called a **statistical** experiment.

Definition: The **sample space** is the set of all possible outcomes of a statistical (random) experiment and is denoted by $\boldsymbol{S}$.

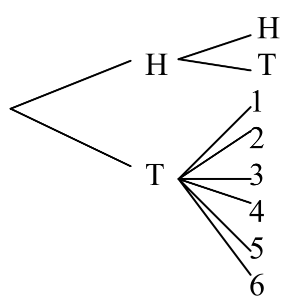
Note: The outcomes are called the **elements** or the **members** of the sample space or simply the **sample points**.

Note: The sample space can be;

a. Finite and countable, e.g. $\boldsymbol{S}=\{1,2,3,4,5,6\}$ where $\#\boldsymbol{S}=6$.
b. Infinite and countable, e.g. $\boldsymbol{S}=\{H,TH,TTH,TTTH,\ldots\}$.
c. Infinite and uncountable, e.g. $\boldsymbol{S}=[0,1]$.

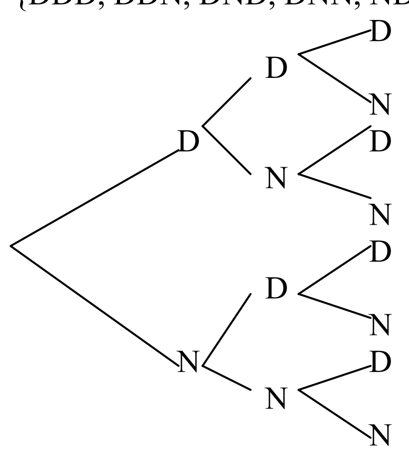
Definition: The **tree diagram** is a method to describe the sample space.

Ex.1 (2.2/23): $\boldsymbol{S}=\{HH, HT, T1, T2, T3, T4, T5, T6\}$



Ex.2 (2.3/23): If D: Defective and N: Non-defective then,

$\boldsymbol{S}=\{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$ and,



Note: Another method of describing a sample space with large or infinite number of elements is by a statement or a rule.

Ex.3: (2/29), $\boldsymbol{S}=\{(x, y) \mid x^2 + y^2 \leq 9, x \geq 0, y \geq 0\}$.

2.2 Events

Objectives:

1. **To define the event and the disjoint events.**
2. **To identify the different types of events**
3. **To define the operations on events.**
4. **To describe an event by Venn diagrams.**
5. **To define the permutations and combinations of outcomes.**

Definition: An **event** is a subset from the sample space.

Ex.1: If a die is rolled once, then write ***S***, the events A, and B where A: the outcome is an odd number & B: the outcome is a number more than 4.

Types of events

1. **Simple**: It consists of ONLY one element of ***S***.

Ex.2: If ***S***={1,2,3,4,5,6}, A={1} and B={3}.

2. **Compound**: It consists of more than one element of ***S***.

Ex.3: If ***S***={a,b,c,d,e}, A={a,b}, and B={a,c,e}.

3. **Impossible**: It contains NO elements at all, and is called the null event and is denoted by φ.

Ex.4: If C={x| x is an even factor of 7} then C= φ.

4. **Sure**: It simply contains all the elements of ***S***.

Definition: The **complement** of an event A, denoted by A', is the set of all elements that are in ***S*** but not in A.

Ex.5: If ***S***={1,2,3,4,5,6,7,8} and A={1,2,3,7} then find A'.

Definition: The **intersection** of two events A & B, denoted by A∩B, is the set of all elements that are common to A & B.

Ex.6: If ***S***={1,2,3,4,5,6,7,8}, A={1,2,3,7}, B={6,7,8}, and C={3,4,5} then find A∩B & B∩C.

Definition: Two events A & B are said to be **mutually exclusive**, or **disjoint**, if and only if (**iff**) A∩B=φ.

Definition: The **union** of two events A & B, denoted by A∪B, is the set of all elements that belong to A, B, or both.

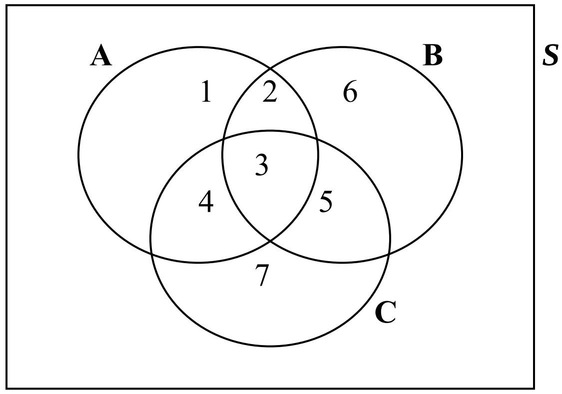
Ex.7: If A={1,2,3,7}, B={6,7,8}, and C={3,4,5} then find A∪B & B∪C.

Ex.8: Solve (16/30) given that ***S***={x| 0< x <12}, M={x| 1< x <9}, and N={x| 0 < x < 5}.

Definition: **Venn Diagrams** is a method of describing the events graphically.

Ex.9: Consider the Venn diagram below, then shadow the following:

1. A∩B (regions 2 &3) 2. B∩C (regions 3 &5)
3. A∪C (regions 1, 2, 3, 4, 5, & 7) 4. B'∩A (regions 1 & 4)
5. A∩B∩C (region 3) 6. (A∪B)∩C' (regions 1, 2, & 6)



Results:

1. $A \cap \varphi = \varphi$ 2. $A \cup \varphi = A$ 3. $A \cap A' = \varphi$
4. $A \cup A' = \boldsymbol{S}$ 5. $\boldsymbol{S}' = \varphi$ 6. $\varphi' = \boldsymbol{S}$
7. $(A')' = A$ 8. $(A \cap B)' = A' \cup B'$ 9. $(A \cup B)' = A' \cap B'$

Ex.10: Solve (5/29), given that a die to be rolled then either a coin to be flipped once, for an even number, or twice, for an odd number.

Ex.11: Solve (9/30) referring to the last example.

Theorem: The number of **permutations** of n distinct objects taken r at a time is: ${}_nP_r = \frac{n!}{(n-r)!}$

Theorem: The number of **combinations** of n distinct objects taken r at a time is: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Ex.12: If 5 persons A, B, C, D, and E are candidates for three jobs. Find the number of ways to select 3 persons for the positions of:

a. Three clerks.

b. A chairman, a vice chairman, and a secretary.

2.4 Probability of an Event

Objectives:

1. **To define the probability of an event.**
2. **To define equally likely sample spaces.**
3. **To define the experimental probability and its relation to the relative frequency.**

Definition: The **probability** of an event A, denoted by **P(A)**, is the sum of the weights of all points in A.

Properties:

1. $0 \le P(A) \le 1$ 2. $P(\boldsymbol{S}) = 1$ 3. $P(\varphi) = 0$
4. If $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

Ex.1 (2.22/40): A coin is tossed twice, what is the probability that at least one head occurs? Assume all outcomes have the same weight. ($P(A) = \frac{\#A}{\#S}$ for equally likely outcomes of $\boldsymbol{S}$)

Ex.2 (2.23/40): P(Even) = 2P(Odd), find $P(i)$: $i = 1, 2, \ldots, 6$.

Ex.3 (2.24/41): Refer to last example. A: an even number and B: a number divisible by 3. Find P(A∩B) and P(A∪B).

Theorem: If the sample space ***S*** consists of N equally likely outcomes, and if an event A consists of n elements of ***S***, then: $P(A) = \frac{\# A}{\# S} = \frac{n}{N}$

Ex.4 (2.25/41): I: 25, M: 10, E: 10, and C:8. Find P(I) and P(C or E).

Ex.5 (2.26/42): From a hand of five cards, find the probability of getting 2 aces and 3 jacks.

Definition: The **subjective** probability is arrived to using intuition, personal beliefs, or other indirect information.

Definition: The **objective** probability is arrived to by calculating the relative frequency of an event if the random experiment is done large number of times.

Ex.6: Consider the following table

Outcome	**N**	**f**	**rf**
Head	1,000	490	0.49 ≈ 0.5
Head	10,000	5010	0.501 ≈ 0.5
Head	100,000	49,900	0.499 ≈ 0.5

Note: The objective probability = theoretical probability when N becomes sufficiently large, $N \to \infty$.

Ex.7: Solve problem (13/47).

2.5 Additive Rules of Probability

Objective:

To introduce the additive rules of calculating probability

Theorem: If A & B are any two events, then

P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

Corollary: If A & B are mutually exclusive, then

P(A ∪ B) = P(A) + P(B)

Corollary: If A_1, A_2, A_3, …, A_n are mutually exclusive events, then

$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$

Corollary: If A_1, A_2, A_3, …, A_n constitutes a partition of the sample space ***S***, then

$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$
$= P(S) = 1.$

Theorem: If A, B, and C are three events then

P(A ∪ B ∪ C) = P(A) + P(B) + P(C) – P(A ∩ B) – P(A ∩ C)
– P(B ∩ C) + P(A ∩ B ∩ C)

Ex.1 (2.27/44): If P(A) = 0.8, P(B) = 0.6, and P(A ∩ B) = 0.5, then find the probability of at least one offer.

Ex.2 (2.28/44): If A: Total is 7, B: Total is 11, then find the probability of getting a total of 7 or 11.

Ex.3 (2.29/45): If G: Green, W: White, R: Red, and B: Blue is 11, then find P(G ∪ W ∪ R ∪ B).

Theorem: Since A & A' are complementary events, then

P(A ∪ A') = P(A) + P(A') = 1 = P(***S***) ➔ P(A') = 1 – P(A)

Ex.4 (2.30/45): If the probabilities of 3, 4, 5, 6, 7, and 8 or more are 0.12, 0.19, 0.28, 0.24, 0.1, and 0.07 respectively, then find P(E) where E: at least 5.

Ex.5: Solve problem (5/46) given that P(M) = 0.7, P(B) = 0.4, and P(M∪B) = 0.8, then find P(M and B) and P(neither city).

2.6 Conditional Probability

Objective:

1. **To define the conditional probability.**
2. **To define the independence of events.**

Definition: The probability of an event A given that another event B has occurred is called the **conditional probability** and is given by: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, $P(B) > 0$

Ex.1 (2.31/49): If P(A) = 0.82, P(D) = 0.83, and P(A ∩ D) = 0.78, then find P(A|B) and P(B|A).

Note: The probability that an event **A occurs and not B** is given by:
$P(A \cap B') = P(A - B) = P(A) - P(A \cap B)$

Definition: If the occurrence of an event A does not affect the occurrence of another and vice versa, then the two events A & B are said to be independent, and P(A|B) = P(A) and P(B|A) = P(B).

Ex.2: Solve problem (2/54). Let J: Junior, S: Senior, and G: Graduate. Also, let A: Grade A and N: Not grade A.

Ex.3: Solve problem (4/54). Let NS: Non-smoker, MS: Moderate smoker, HS: Heavy smoker, H: Hypertension, and N: No hypertension.

2.7 Multiplicative Rules of Probability

Objective:

1. **To introduce the additive rules of calculating probability**
2. **To generalize the rule considering the independence.**

Theorem: If A & B are NOT mutually exclusive then
$P(A \cap B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)$

Ex.1 (2.32/50): Let D_1: First fuse is defective and D_2: Second fuse is defective. Find P(D_1 and D_2)

Ex.2 (2.33/51): Let B_1: First ball is black, B_2: Second ball is black, W_1: First ball is white, and W_2: Second ball is white. $P(B_2)$ =?

Theorem: The two events A and B are statistically **independent** ↔ $P(A \cap B) = P(A) \times P(B)$

Ex.3 (2.34/52): Let F: Fire engine is available, ➔ P(F) = 0.98 and let A: Ambulance is available, ➔ P(A) = 0.92 then find P(Both are available).

Note: If A and B are two independent events, then:

1. A & B'   2. A' & B   3. A' & B'

are all independent.

Proof: $P(A \cap B') = P(A - B) = \dots$

Ex.4 (2.35/52): The system works if the components A & B are both working and if C or D is working.

Theorem: 1. If in a certain experiment $A_1, A_2, A_3, \dots, A_k$ can occur, then $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{k-1})$.

2. If $A_1, A_2, A_3, \dots, A_k$ are independent then,
$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdots P(A_k)$.

Ex.5 (2.36/53): If 3 cards are drawn without replacement then find the probability that the three cards are a red ace, a 10 or a Jack, and a card greater than 3 but less than 7, respectively.

Ex.6 (2.37/54): If a biased coin, such that P(H) = 2·P(T), is flipped 3 times then find P(2T & 1H).

Ex.7: Solve problem (19/56). Let A: Aspirin, L: Laxative, and T: Thyroid. Find; a. P(Both are T) b. P(The two tablets are different).

2.8 Baye's Rule

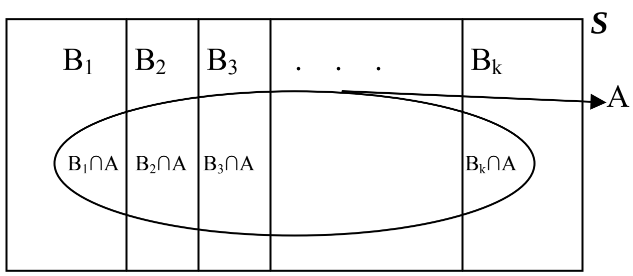
Objective:

1. **To introduce the theorem of total probability.**
2. **To introduce Baye's rule.**

Theorem: (Theorem of **total probability** or the **rule of elimination**)

If $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space ***S*** such that $P(B_i) \neq 0$ for i=1, 2, …, k, then for any event A of ***S***

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(A \mid B_i) P(B_i)$$



Ex.1 (2.38/59): Let D: Product is defective, $P(B_1) = 0.3$, $P(B_2) = 0.45$, and $P(B_3) = 0.25$. If $P(A|B_1) = 0.02$, $P(A|B_2) = 0.03$, and $P(A|B_3) = 0.02$ then find P(Selected product is defective).

Theorem: (**Baye's rule**)

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space ***S*** such that $P(B_i) \neq 0$ for i=1, 2, …, k, then for any event A of ***S***, such that $P(A) \neq 0$

$$P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j) P(B_j)}{\sum_{i=1}^{k} P(A \mid B_i) P(B_i)} \quad \text{for } j = 1,2,\ldots,k$$

Ex.2 (2.39/60): Refer to Ex.1 and find $P(B_3|A)$.

Ex.3: Problem (1/60). Let C: The person has the disease and let D: The doctor diagnoses the disease correctly. P(C) = 0.05, P(C') = 0.95, P(D|C) = 0.78, and P(D|C') = 0.06 ➔ P(C) = ?

Ex.4: Problem (3/60). Refer to Ex.3 and find P(C|D).