# Chapter 5
# Some Discrete Probability Distributions

## 5.3 The Binomial Distribution

**Objectives:**

1. **To define the Bernoulli process and the binomial distribution.**
2. **To find the probabilities of a binomial r.v.**
3. **To calculate μ and $\sigma^2$ of a binomial r.v.**

*Definition*: (Bernoulli Process (Experiment)) It is any experiment that satisfies the following;

1. The experiment consists of **n** repeated trials.
2. Each trial has **only** two outcomes, namely a **S**uccess or a **F**ailure.
3. The probability of the Success, denoted by ***p***, is **fixed**, sampling with replacement, during all the trials.
4. The repeated trials are **independent**.

*Definition*: (Binomial Distribution) If $X$: the **number of successes in a Bernoulli process** then $X$ is called a **binomial r.v.** with two parameters ***n***: the number of trials and ***p***: the probability of success, $q = 1 - p$. The probability of getting $x$ successes is given by;

$$f(x) = P(X = x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x} \ ; \ x = 0,1,\ldots,n,$$

which is the pdf of $X$.

**Ex.1 (5.4/119)**: If $n = 4$ and $p = 3/4$ , then find $P(X = 2)$.

*Definition*: (Binomial Cumulative Distribution Function) If $X$ is a binomial r.v. then the cdf of $X$ is given by; $F(x) = P(X \le x) = \sum_{t=0}^{x} b(t; n, p) = B(x; n, p)$, for which certain numerical values are given in table A.1 on pp. 661-666 for some values of $n$ & $p$.

**Ex.2 (5.5/120)**: If $n = 15$ and $p = 0.4$ , then find;

a. $P$(At least 10). b. $P$(Between 3 & 8 inclusive).
c. $P$(Exactly 5). d. $P$(At most 7). e. $P$(Less than 10).

**Ex.3 (5.6/120)**: Let $X$: # of defective items and let $Y$: # of shipments with at least one defective items among 20 tested.

a. If $n = 20$ and $p = 0.03$, then find $P(X \ge 1)$.
b. If $n = 10$ and $p = P(X \ge 1)$, then find $P(Y = 3)$.

*Theorem*: If $X$ is a binomial r.v. with a pdf $b(x;n,p)$ then;

1. $\mu = E(X) = np$ 2. $\sigma^2 = V(X) = npq$.

**Ex.4**: Refer to Ex.2 and find μ and $\sigma^2$.

**Ex.5**: Problem (12/125). Let $W$: # of cars out of state, then $n$=9 & $p$=1/4.

**Ex.6**: Problem (15/125). Let $Z$: # of mice contracting the disease, then $n$=5 & $p = 0.6$.

## 5.4 The Hypergeometric (HG) Distribution

**Objectives:**

1. **To define the HG distribution and calculate its probabilities.**

2. **To recognize the relationship with the binomial r.v.**
3. **To calculate μ and $\sigma^2$ of a HG r.v.**

*Definition*: (Hypergeometric Experiment) It is any experiment done **without replacement** and satisfies the following;

1. A random sample of size $n$ is selected from $N$ items.
2. A group of $k$ items of the $N$ items may be classified as successes and $N - k$ failures.

*Definition*: (Hypergeometric Distribution) If $X$: the **number of successes in a sample of size *n* selected from a group of *N* items**, then $X$ is called a **HG r.v.** with three parameters ***N***, ***n***, and ***k***. The probability of getting $x$ successes is given by;

$$P(X=x)=h(x;N,n,k)=\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}\ ;\ \max\{0,n-(N-k)\}\le x\le \min\{k,n\}$$

which is the pdf of $X$.

**Ex.1 (5.12/128)**: $N = 40$, $k = 3$, and $n = 5$. Find $P(X = 1)$.

*Theorem*: The **mean** and **variance** of a HG distribution is $\mu=E(X)=\frac{nk}{N}$ and $\sigma^2=V(X)=\frac{N-n}{N-1}n\frac{k}{N}\left(1-\frac{k}{N}\right)$ respectively.

**Ex.2 (5.13/129)**: Referring to Ex.1, find μ and $\sigma^2$ for the r.v. $X$.

*Relationship To The Binomial Distribution*

1. HG probabilities can be approximated by binomial probabilities when $\frac{n}{N}\le 0.05$ for a good approximation.
2. Note that $\mu=E(X)=np=\frac{nk}{N}$ and $\sigma^2=V(X)=n\frac{k}{N}\left(1-\frac{k}{N}\right)$ for the binomial distribution with the difference of the **correction factor** $\frac{N-n}{N-1}\to 1$ as $n$ gets sufficiently small relative to $N$.

**Ex.3 (5.14/129)**: $N$=5000, $k$=1000, and $n$=10. Find $P(X=3)$. Note that the exact probability is $h(3; 5000, 10, 1000) = 0.2015$. What is the expected number of blemished tires in the sample of 10 tires purchased from the shipment?

**Ex.4**: Problem (1/131). **)**: $N$=52, $k$=12, and $n$=7.

**Ex.5**: Problem (18/132). **)**: $N$=9, $k$=?, $n$=5, G=3, B=2, and R=4. Find $P$(Both blue balls & at least 1 red ball).

## 5.5 The Geometric Distribution

## Objectives:

1. **To define the geometric distribution.**
2. **To find the probabilities of a geometric r.v.**
3. **To calculate μ and $\sigma^2$ of a geometric r.v.**

*Definition*: If repeated trials can result in a **S**uccess with probability $p$ and a **F**ailure with probability $q=1-p$, then the probability distribution of the r.v. $X$: **the number of trials <u>on which the</u> (needed to) first Success occur**, is $g(x;p)=pq^{x-1}, x=1,2,...$

**Ex.1 (5.17/134)**: If $p = 1/100 = 0.01$, then find $P(X = 5)$.
**Ex.2 (5.18/135)**: If $p = 1/20 = 0.05$, then find $P(X = 5)$.

*Theorem*: The **mean** and **variance** of a geometric distribution is $\mu_X = E(X) = \frac{1}{p}$ and $\sigma_X^2 = V(X) = \frac{1-p}{p^2} = \frac{q}{p^2}$ respectively.

**Ex.3**: Refer to Ex.1 and find the mean, *the average number of tests needed to get the first defective item*, and the variance.

## 5.6 Poisson Distribution and Poisson Process

## Objectives:

1. **To define the Poisson r.v. and the Poisson process.**
2. **To find the probabilities of a geometric r.v.**
3. **To calculate μ and $\sigma^2$ of a geometric r.v.**
4. **To use the Poisson distribution to approximate the binomial probabilities.**

*Definition*: If the r.v. $X$: **the number of outcomes occurring during a Poisson process**, usually denoted by **t**, then $X$ has a Poisson distribution with a pdf; $p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ , $x = 0, 1, 2,...$

where $\boldsymbol{\lambda}$: Average number of outcomes per unit time or region and $\boldsymbol{t}$: Length of the time interval or the size of the region.

*Properties of the Poisson process*

1. The number of outcomes in any interval is independent of the other disjoint intervals. Poisson process has no memory.
2. The probability that a single outcome will occur in a very short time interval is proportional to the length of the interval.
3. The probability that more than one outcome will occur in such a very short time interval is negligible.

*Note*: Certain values of the Poisson probability sum (**Poisson cdf**), i.e. $F(x) = P(X \le x) = \sum_{t=0}^{x} p(t;\lambda t) = P(x;\lambda t)$, are given in table A.2 on pp. 667-669 for some values of $\lambda t = \mu$.

**Ex.1 (5.19/137)**: If Ave. = 4/1millisecond, then find $P(X = 6)$. $\lambda t = 4$.
**Ex.2 (5.20/137)**: If $\lambda t = 10$, then find $P(X > 15)$.

*Theorem*: Each of the **mean** and **variance** of a Poisson distribution is $\lambda t$.

**Ex.3**: Problem (3/139). $\lambda = 5$, $t = 1$.
**Ex.**4: Problem (10/139). $\lambda = 6$, $t = 1$.

*Theorem*: Let $X$:B($n$,$p$), then as $n \to \infty, p \to 0$ and $\lambda t = \mu = np$ remains constant, b($x$; $n$, $p$) → p($x$; $\mu$).

**Ex.5 (5.21/138)**: If $p = 0.005$, $n = 400$, then find $P(X = 1)$.