Chapter 6
Some Continuous Probability Distributions

6.1 The Continuous Uniform Distribution

Objectives:

1. **To define the continuous uniform distribution.**
2. **To find the probabilities of a continuous uniform r.v.**
3. **To calculate μ and σ^2 of a continuous uniform r.v.**

Definition: The density function of the continuous **uniform** r.v. X on the interval [A,B], denoted by X:U(A,B), is given by:

$$f(x) = \begin{cases} \frac{1}{B-A} & , A \le x \le B \\ 0 & , e.w. \end{cases}$$

and it has a rectangular shape.

Ex.1 (6.1/143): X:U(0,4), then find the pdf of X and $P(X \ge 3)$.

Theorem: If X:U(A,B) then : $\mu_X = \frac{A+B}{2}$ and $\sigma^2 = \frac{(B-A)^2}{12}$.

Ex.2: Refer to Ex.1 and find the mean and the variance.

6.2 The Normal Distribution

Objectives:

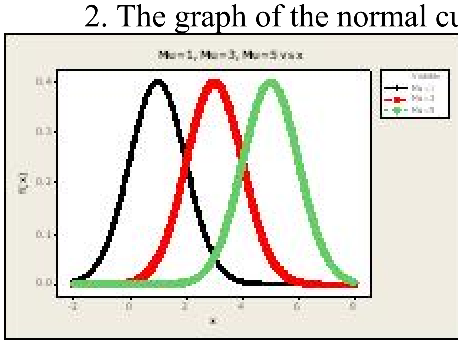
1. **To define the normal distribution.**
2. **To identify the properties of the normal curve.**

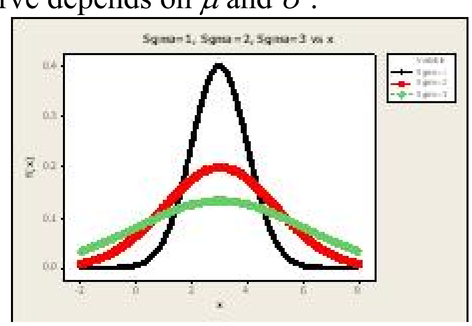
Definition: The density function of the **normal** r.v. X, denoted by X:N(μ,σ^2), is given by: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < +\infty$

where $-\infty < \mu < +\infty$ and $0 \le \sigma^2$.

Note: 1. The curve of this pdf is called the normal curve, and X is said to have the **normal** distribution (Gaussian dist.).

2. The graph of the normal curve depends on μ and σ^2.





Properties of the normal curve:

1. Mean = Median = Mode = μ.
2. The curve is symmetric about μ and bell shaped.
3. The area under the curve is 1.
4. The mean = μ and the variance = σ^2.

6.3 Areas Under The Normal Curve

Objectives:

1. **To define the standard normal distribution.**
2. **To calculate areas under the curve of the standard normal r.v.**
3. **To calculate areas under the curve of a general normal r.v.**
4. **To find values of a normal r.v. corresponding to given areas.**

Definition: If $x_1 < x_2$ are two different real numbers, then the area of the region between x_1 and x_2, given that X:N(μ , σ^2), is given by

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma^2)dx = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

And since this integral can NEVER be found in closed form and need to be calculated for each different value of μ and σ, so we define the **normal score** or the **z-score** for any normal value by the formula $Z = \frac{X-\mu}{\sigma}$. The transformed r.v. Z has the **standard normal** distribution and Z:N(0,1). Tables for calculated areas, i.e. cumulative probabilities, under the standard normal curve and less than a given point z are given in Appendix A3 on pp. 670 & 671. The table gives probabilities of the form $P(Z < z)$.

Cases of finding areas

1. Areas below the z value or to the left of the z value:
   $P(Z < z)$ = read directly from the table.
2. Areas above the z value or to the right of the z value:
   $P(Z > z) = 1 - P(Z < z) = P(Z < -z)$.
3. Areas between two z values: $P(z_1 < Z < z_2) = P(Z < z_2) - P(Z < z_1)$.

Ex.1: Given that Z:N(0,1) then find the following:

a. The area below 1. b. The area to the left of – 2.26.
c. $P(Z < 0.05)$ d. $P(Z > 0.5)$
e. $P(Z \geq -1.82)$ e. The area between – 2.39 and – 1.
f. $P(-0.98 < Z < 0.35)$ g. $P(1.15 < Z < 2.01)$

Ex.2 (6.3/149): Find k such that:

a. $P(k < Z) = 0.3015$ b. $P(k < Z < -0.18) = 0.4197$

Ex.3: Problem (4/156), given that X:N(30 , 36) find the following:

a. The area to the right of $x = 17$.
b. The area to the left of $x = 22$.
c. The area between $x = 32$ and $x = 41$.
d. The value of X that has an area of 80% to its left.
e. The two values of X that contain the **middle** 75% of the area.

6.4 Application Of The Normal Distribution

Objective: To solve application problems of the normal distribution.

Ex.1: Problem (7/157). Let X:N(40 , $(6.3)^2$) where X: Mouse lifetime.
Ex.2 (6.9/153): Let Y:N(3 , 2.5×10^{-5}) where Y: Ball bearing diameter.
Ex.3 (6.10/154): Let U:N(1.5 , 0.04) where U: Measurements.
Ex.4 (6.11 & 12/154 & 155): Let V:N(40 , 4) where V: Resistance of electrical resistors.
Ex.5 (6.13/155): Let W:N(74 , 49) where W: Exam grade.
Ex.6 (6.14/156): Refer to Ex.5 and find the 6^{th} decile.
Ex.7: Problem (19/158). Let X:N(115 , 144) where X: IQ.

6.5 The Normal Approximation To The Binomial

Objective: To use the normal distribution to approximate the binomial distribution.

Definition: If X:B(n,p) then for large n, as $n \to +\infty$, the limiting form of the distribution of $Z = \frac{X - np}{\sqrt{npq}}$ is standard normal N(0 , 1), X~N(np , npq).

Note: 1. The approximation is good if n is large or p is close to 0.5.
2. The approximation is good if $np \geq 5$ and $nq \geq 5$.

Continuation Correction

1. $P(a \leq X \leq b) = P(a - 1/2 \leq X \leq b + 1/2)$.
2. $P(a < X) = P(a + 1 \leq X) = P(a + 1 - 1/2 \leq X)$.
3. $P(X < b) = P(X \leq b - 1) = P(X \leq b - 1 + 1/2)$.
4. $P(X = c) = P(c - 1/2 \leq X \leq c + 1/2)$.

Ex.1 (6.15/162): If $n = 100$ and $p = 0.4$ then find $P(X < 30)$.
Ex.2 (6.16/162): If $n = 80$ and $p = 0.25$ then find $P(25 \leq X \leq 30)$.
Ex.3: Problem (3/164) If $n = 100$ and $p = 0.01$ then find $P(X < 1)$.

6.6 Gamma And Exponential Distributions

Objectives:

1. **To define the gamma function and the gamma distribution.**
2. **To identify a special case of the gamma r.v., the exponential.**
3. **To find the probabilities of a gamma and exponential r.v.'s.**
4. **To calculate μ and σ^2 of a gamma and exponential r.v.'s.**
5. **To identify the relationship to the Poisson r.v.**

Definition: The **gamma** function is defined as a function of a constant α:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \ , \ \ \alpha > 0$$

Note:

1. For $\alpha > 1$, the recursive relation is satisfied: $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
2. For a positive integer n: $\Gamma(n) = (n-1)!$.
3. $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition: Let X be a r.v. that has a **gamma distribution** with parameters α and β, denoted by X:$\Gamma(\alpha , \beta)$, then the pdf of X is given by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & , \ x > 0 \\ 0 & , \ e.w. \end{cases}$$ where $\alpha, \beta > 0$.

Definition: When $\alpha = 1$ we have the special case of the gamma distribution which is the **exponential distribution**, denoted by X:Exp(β), with a parameter β and has a pdf:

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & , \ x > 0 \\ 0 & , \ e.w. \end{cases}.$$

Theorem: The mean and variance of the gamma distribution are:

1. $\mu = \alpha\beta$ 2. $\sigma^2 = \alpha\beta^2$.

Corollary: The mean and variance of the exponential distribution are:

1. $\mu = \beta$ 2. $\sigma^2 = \beta^2$.

Relationship to Poisson process:

The **length of the segments of time or space** occurring until some specific number, **α**, of events has occurred is a r.v. having a gamma distribution with parameters α and λ = 1/β. The exponential distribution describes the time between successive events in a Poisson process.

6.7 Applications of Gamma And Exponential Distributions

Objective: To solve application problems of the gamma and exponential distributions.

Note: The mean of exponential distribution is $\mu = \beta \Rightarrow \frac{1}{\beta} = \lambda$ in Poisson distribution where β is called the mean time between failures.

Ex.1 (6.17/168): T: Exp(5) and $n = 5$ then find $P(X \geq 2)$.

Ex.2 (6.18/169): If $\lambda = 5$ calls/min. and X:Γ(2 , 5) then find $P(X \leq 1)$.

Ex.3: Problem (7/174) T: Exp(4), $n = 6$, and $p = P(T<3)$. Find $P(X \geq 4)$.

6.8 Chi-squared Distribution

Objectives:

1. **To define the chi-squared distribution.**
2. **To calculate μ and σ^2 of a chi-squared r.v.**

Note: It is a special case of the gamma distribution when $\alpha = \nu/2$ and $\beta = 2$, where ν is called the **degrees of freedom**.

Definition: Let X be a r.v. that has a **Chi-squared distribution** with ν degrees of freedom, denoted by X: χ^2_ν, then the pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} x^{\frac{\nu}{2}-1} e^{-x/2} & , x > 0 \\ 0 & , e.w. \end{cases}$$ where $\nu > 0$.

Theorem: The mean and variance of the chi-squared distribution are:

1. $\mu = \nu$ 2. $\sigma^2 = 2\nu$.