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Ph.D. Dissertation Proposal

Particular Overrings of PVMDs
t-Regularity in Integral Domains

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Contents

1 Abstract

This thesis will investigate special overrings of Prüfer v-multiplication domains in an attempt to establish analogues for well-known results on overrings of Prüfer domains. Here algebraic objects such as ideals (resp., prime ideals, maximal ideals) are expected to cede their places to t-ideals (resp., prime t-ideals, maximal t-ideals). The t-class semigroup of an integral domain is the semigroup of the isomorphy classes of the (integral) t-ideals with the operation induced by ideal t-multiplication. This thesis will also study ring-theoretic conditions of an integral domain that reflect in the Clifford property or Boolean property of its t-class semigroup. Particularly, we aim to characterize Boole t-regularity for GCD domains, extending Kabbaj and Mimouni's 2003 and 2007 results on Bezout domains and PVMDs.

2 Introduction

Let R be an integral domain and K its quotient field. Let I and J be two nonzero fractional ideals of R. We define the fractional ideal $(I : J) := \{x \in K \mid xJ \subseteq I\}$. We denote $(R : I)$ by I^{-1} and $(I^{-1})^{-1}$ by I_v. We say that I is divisorial or a v-ideal of R if $I_v = I$. We define $I_t = \bigcup\{J_v \mid J \subseteq I \text{ finitely generated}\}$. The ideal I is a t-ideal if $I_t = I$. Under the operation $(I, J) \to (IJ)_t$, the set of t-ideals of R is a semigroup with unit R. An invertible element for this operation is called a t-invertible t-ideal of R. For more details about these basic notions, see [13]. The class semigroup of an integral domain R, denoted $S(R)$, is the (multiplicative Abelian) semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals. By analogy, the t-class semigroup of an integral domain R, denoted $S_t(R)$, is the (multiplicative Abelian) semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals. We'll deal with ring-theoretic properties of an integral domain R that reflect reciprocally in the Clifford or Boolean property of $S_t(R)$.

A commutative semigroup S is said to be a Clifford semigroup if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x^2a = x$. A semigroup S is said to be Boolean if for each $x \in S, x = x^2$. Clearly, a Boolean semigroup is a Clifford semigroup. The structure of commutative Clifford semigroups is very closed to that of groups. For, every commutative Clifford semigroup is a disjoint union of subgroups G_e, where e ranges over the set of idempotent elements of S and G_e is the largest subgroup of S with identity equal to e [17]. The G_es are called the constituent groups of S.

An ideal I of R is said to be L-stable (here L stands for Lipman) if $R^I = \bigcup(I^n : I^n) = (I : I)$, and R is called an L-stable domain if every nonzero ideal of R is L-stable. As in [19], an ideal I of R is said to be a stable (resp., strongly stable) ideal if I is invertible (resp., principal) in $(I : I)$, and R is called a stable (resp., strongly stable) domain if every nonzero ideal of R is stable (resp., strongly stable). Recall that a stable domain is L-stable[2] and a strongly stable domain is obviously stable.

In 1994, Zanardo and Zannier [21] proved that if R is an integrally closed domain and $S(R)$ is a Clifford semigroup then R is a Prüfer domain. In 1996, Bazzoni and Salce [7] investigated the structure of the class semigroup for a valuation domain V, stating that $S(V)$ is a Clifford semigroup. In [3] and [4], Bazzoni showed that a Prüfer domain of finite character have Clifford class semigroup. In 2001, Bazzoni [6] gave a complete characterization of integrally closed domains with Clifford class semigroup. In 2003, Kabbaj and Mimouni [19] proved that "*If R is an integrally closed domain then: R is a strongly stable domain if and only if R is a strongly discrete Bezout domain of finite character. Moreover, if any one of these conditions holds then R has Boole regular class semigroup.*" In 2007, Kabbaj and Mimouni [20] established t-analogues of basic results on t-regularity. They proved that a Krull domain (resp., UFD) has Clifford (resp., Boole) t-class semigroup. Their main result states that "*a PVMD has Clifford t-class semigroup if and only if it is a Krull-type domain.*"

One of the objectives of this thesis is to seek ring-theoretic conditions of an integral domain that reflect reciprocally in the Clifford property or Boolean property of its t-class semigroup. Particularly, we will study Boole t-regularity in GCD domains, trying to extend Kabbaj-Mimouni's 2003 and 2007 results on Bezout domains and PVMDs.

Let I be an ideal of R. We define the algebraic objects: $T(I) = \cup(R : I^n)$ and $\Omega(I) = \{u \in K : ua^{n(a)} \in R,\ a \in I$ and $n(a)$ some positive integer$\}$. We have $(I : I) \subseteq R^I \subseteq T(I) \subseteq \Omega(I)$ and $(I : I) \subseteq I^{-1} \subseteq T(I) \subseteq \Omega(I)$. Notice that $\Omega(I)$ is a variant of the Nagata transformation $T(I)$, and useful in the case when I is not finitely generated. If I is a finitely generated ideal, then $\Omega(I) = T(I)$. It is worthwhile noting that $\Omega(I)$, $T(I)$, $(I : I)$ and R^I are overrings of R for each ideal I in the domain R. If R is a Prüfer domain, all these structures are well described. The other main objective of this thesis is to describe these objects for PVMDs.

3 Literature Review

In 1968, Brewer [8] gave the description of $\Omega(I)$ for each ideal in integral domains. He proved that "*if I is a nonzero ideal of R, then $\Omega(I) = \cap R_P$, where P varies over the set of prime ideals not containing I*". Gilmer [13] described $T(I)$ of an ideal in a Prüfer domain which is contained in a finite number of minimal prime ideals, specifically, "*let R be a Prüfer domain, I a nonzero ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, and $\{M_\beta\}$ the set of maximal ideals that do not contain I. Then (1) $T(I) \subseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$, where Q_α is the unique prime ideal determined by $\bigcap_{n=1}^{\infty} I^n R_{P_\alpha} = Q_\alpha R_{P_\alpha}$; (2) If the set $\{P_\alpha\}$ is finite, equality holds.*"

Fontana, Huckaba, and Papick [12] studied the endomorphism ring of an ideal in Prüfer domains: "*Let R be a Prüfer domain, I a nonzero ideal of R, $\{Q_\alpha\}$ the set of maximal prime ideals of $Z(R, I)$, and $\{M_\beta\}$ the set of maximal ideals that do not contain I. Then $(I : I) \supseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$. If R is a QR-domain, equality holds*". Note that R^I coincides with $(I : I)$ in a Prüfer domain [2].

Many papers in the literature deal with the fractional ideal I^{-1}. The main problem is to examine settings in which I^{-1} is a ring. Huckaba and Papick [18] stated the following: "*Let R be a Prüfer domain, I a nonzero ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, and $\{M_\beta\}$ the set of maximal ideals that do not contain I. Then $I^{-1} \supseteq (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$. If I^{-1} is a ring, equality holds.*" Also the equality $I^{-1} = (I : I)$ has been investigated in various contexts of integral domains.

Recall that an integral domain R is Clifford regular (resp., Boole regular) if $S(R)$ is a Clifford (resp., Boolean) semigroup. By analogy, an integral domain R is Clifford (resp., Boole) t-regular if $S_t(R)$ is a Clifford (resp., Boolean) semigroup. Clearly, a Boole (t-)regular domain is Clifford (t-)regular.

In 2001, Bazzoni [6] gave the complete structure of integrally closed Clifford regular domains; specifically, "*an integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character.*" In 2003, Kabbaj and Mimouni [19] investigated Boole regularity for integrally closed domains; namely, "*if R is an integrally closed domain then: R is a strongly stable domain if and only if R is a strongly discrete Bezout domain of finite character. Moreover, when any one condition holds, R is a Boole regular domain and $S(R) = \mathcal{F}ov(R)$, where $\overline{T}$ is identified with T for each fractional overring T of R.*" They established transfer results for Clifford t-regularity and Boole t-regularity in Noetherian and Noetherian-like settings as well as in various contexts of pullback constructions.

In 2007, Kabbaj and Mimouni [20] studied ring-theoretic properties of an integral domain which reciprocally reflect in semigroup-theoretic properties of its t-class semigroup. Among basic results, they proved that a Krull (resp., UFD) is always a Clifford (resp., Boole) t-regular domain. They also showed that a pseudo-valuation domain R is always a Clifford t-regular doamin; moreover, R is Boole t-regular if and only if it is issued from a strongly discrete valuation ring. Their main result recovers and generalizes Bazzoni's theorem [6, Theorem 4.5] (cited above) to the class of PVMDs; namely, "*a PVMD is Clifford t-regular if and only if it is a Krull-type domain.*"

4 Objectives

The two main objectives of this thesis are:

- Investigate particular overrings of Prüfer v-multiplication domains (PVMDs)
- Study t-regularity in some integral domains (including GCDs)

4.1 Particular Overrings of PVMDs

In 2000, Houston, Kabbaj, Lucas, and Mimouni [16] studied $(I : I)$, I^{-1}, and related algebraic objects in the setting of PVMDs. Among others, they proved the following result: "*Let R be a PVMD, I a nonzero ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, $\{Q_\beta\}$ the set of minimal prime ideals of I_t and $\{M_\gamma\}$ the set of maximal t-ideals that do not contain I. Then: I^{-1} is a ring if and only if $I^{-1} = (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\gamma})$ if and only if $I^{-1} = (\bigcap R_{Q_\beta}) \cap (\bigcap R_{M_\gamma})$.*" They also stated a similar result for $(I : I)$.

Our purpose here is to establish analogous results for other algebraic objects related to the t-structure of a given ring. We particularly plan to extend well-known results on Prüfer domains to PVMDs, where the ideals (resp., prime ideals, maximal ideals) are expected to cede their places to t-ideals (resp., prime t-ideals, maximal t-ideals). Our investigation will be guided by the work carried out in [16].

4.2 t-Regularity in Integral Domains

Bazzoni determined equivalent conditions of Clifford regularity for a integrally closed domains and described the structure of its class semigroups [3, 4, 5, 6, 7]. Kabbaj and Mimouni [19] established analogous results for Boole regularity. The main result asserts that "*if R is an integrally closed domain then: R is a strongly stable domain if and only if R is a strongly discrete Bezout domain of finite character; and if any one condition holds, then R is Boole regular.*"

The pseudo-integral closure of a domain R is defined as $\widetilde{R} = \cup(I_t\colon I_t)$, where I ranges over the set of finitely generated ideals of R; and R is said to be pseudo-integrally closed if $R = \widetilde{R}$. Clearly $R' \subseteq \widetilde{R} \subseteq \overline{R}$, where R' and $\overline{R}$ are respectively the integral closure and the complete integral closure of R. In view of [20, Example 2.7], one has to elevate the "integrally closed" assumption in regularity results to "pseudo-integrally closed" in t-regularity. In 2007, Kabbaj and Mimouni [20] recovered and generalized Bazzoni's main theorem [6, Theorem 4.5] to the class of PVMDs; namely, "*a PVMD is Clifford t-regular if and only if it is a Krull-type domain.*"

Our purpose in this section is to characterize t-regularity for pseudo-integrally closed domains. One of our objectives is to establish a Boolean t-analogue and maybe a generalization of the above results for GCD domains.

5 Program of Study

The program comprises four tasks:

TASK 1: Studying book chapters and research papers on the following topics:

- Class semigroups of integral domains
- t-Class semigroups of integral domains
- Clifford and Boole regularity
- Clifford and Boole t-regularity
- Integral closure, complete integral closure, pseudo integral closure
- Overrings of Prüfer domains
- Algebraic objects $\Omega(I)$, $T(I)$, R^I, $(I : I)$, and I^{-1}
- Pullback constructions
- Bezout, PVMD, GCD, Mori, Krull domains

TASK 2: Solving the open problems discussed in the above sections.

TASK 3: Giving a monthly seminar.

TASK 4: Writing the Ph.D. dissertation.

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