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Journal of Algebra 279 (2004) 685–693

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Prüfer-like conditions and pullbacks

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Received 14 December 2003

Available online 26 June 2004

Communicated by Paul Roberts

Abstract

This paper is a continuation of the investigation of some Prüfer-like conditions, such as almost Prüfer, almost Bézout, and almost *GCD* begun by Anderson–Zafrullah and Anderson–Knopp–Lewin, in the context of pullbacks.

1. Introduction

In his study of almost factoriality in integral domains, Zafrullah [11] introduced the notion of almost *GCD*-domain. He defined R to be almost *GCD*-domain (*AGCD*-domain for short) if for $a, b \in R \setminus \{0\}$, there exists a natural number $n = n(a, b)$ with $a^n R \cap b^n R$ principal (or equivalently $(a^n, b^n)_v$ is principal). In [5], Anderson and Zafrullah introduced several classes of integral domains related to almost *GCD*-domains including almost Bézout domains (*AB*-domains), almost Prüfer domains (*AP*-domains), almost valuation domains (*AV*-domains) and others. They showed that if R is *AGCD*-domain, then R', the integral closure of R, is *AGCD*-domain and $R \subseteq R'$ is a root extension, but they left open the question of whether the converse is true. In [2], Anderson, Knopp and Lewin answered this question negatively by constructing an example of a domain R for which R' is *AGCD*-domain and the extension $R \subseteq R'$ is a root extension but R itself is not *AGCD*-domain. Their construction is based on a special case of pullbacks, namely they proved that $R = K + (X, Y)L[X, Y]$, where $K \subseteq L$ is a root extension is the desired example. The purpose of this paper is to continue the study of almost Bézout domains and other

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.04.012

Prüfer-like conditions in pullbacks. In the second section, we deal with the transfer of the notions “almost Prüfer”, “almost Bézout” and “almost *GCD*” to the pullbacks of the diagrams of type ($\square$). We prove that for such diagram, R is an *AP*-domain if and only if T and D are *AP*-domains and $qf(D) = k \subseteq K$ is a root extension (Theorem 2.2). Moreover, if T is local with maximal ideal M, then R is an *AB*-domain (respectively *AGCD*-domain) if and only if T and D are *AB*-domains (respectively *AGCD*-domains, and M is a t-ideal of T) and $qf(D) = k \subseteq K$ is a root extension (Theorems 2.8 and 2.9). As an application, we restate [2, Theorem 3.5] and [5, Theorem 4.9]. The third section deals with more general properties of *AGCD*-domains. We prove that for a domain R and X an indeterminate over R, $R[X]$ is an *AB*-domain if and only if $R[X]$ is an *AP*-domain if and only if R is a field. This leads us, using results of the second section, to characterize when $R = A + XB[[X]]$ (respectively $A + XB[X]$) is an *AB*-domain (respectively *AP*-domain). It turns out that A must be an *AB*-domain (respectively *AP*-domain), $B = K$ is a field and $qf(A) = k \subseteq K$ is a root extension. (For more details on the t and v operations, see [8].)

2. Pullbacks

In this section we examine the transfer of some Prüfer-like properties to pullbacks. To avoid unnecessary repetition, let us fix notation for the rest of this section. Data will consist of a pullback of canonical homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is an integral domain, M is a maximal ideal of T, $\varphi : T \to T/M$ is the natural projection, D is a domain contained in $K = T/M$, and $R = \varphi^{-1}(D)$. We explicitly assume that $R \subset T$ and we shall refer to this as a diagram of type ($\square$). If $k = qf(D) = K$, we shall refer to this as a diagram of type ($\square^*$). The case where $T = V$ is a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V, is of particular interest. We shall refer to this as the classical $D + M$ construction.

Lemma 2.1. *For the diagram* ($\square$)*, if R is an AP-domain* (*respectively AGCD-domain*)*, then the extension $k \subseteq K$ is a root extension* (*respectively $k \subseteq K$ is a root extension and M is a t-ideal of T*)*.*

Proof. Assume that R is an *AP*-domain (respectively *AGCD*-domain). Suppose that $k \subseteq K$ is not a root extension. Then there is $\lambda \in K$ such that $\lambda^n \notin k$ for each natural number n. Set $\lambda = \varphi(a)$ for some $a \in T \setminus M$ and let b be a nonzero (fixed) element of M. Since R is an *AP*-domain (respectively *AGCD*-domain), then there exists a nonzero natural number $r = r(ab, b)$ such that $((ab)^r, b^r)$ is invertible (respectively $((ab)^r, b^r)_v$ is principal). Set $I = ((ab)^r, b^r)$. Then $II^{-1} = R$ (respectively $I_v = cR$). Clearly $I^{-1} = (ab)^{-r}R \cap b^{-r}R$. Let $f \in I^{-1}$ and set $f = (ab)^{-r} f_1 = b^{-r} f_2$, where $f_1, f_2 \in R$. Then $a^r f_2 = f_1$. If $f_2 \notin M$, then $\varphi(f_2) \in D \setminus \{0\}$. So $\lambda^r \varphi(f_2) = \varphi(a^r f_2) = \varphi(f_1)$ and therefore $\lambda^r = \varphi(f_1)/\varphi(f_2) \in k$, which is absurd. Hence $f_2 \in M$ and so $I^{-1} \subseteq b^{-r}M$. For the reverse inclusion, for each $z \in M$ and each $x \in I$, write $x = \alpha(ab)^r + \beta b^r$, where $\alpha, \beta \in R$. Then $(zb^{-r})x = z\alpha a^r + z\beta \in R$ (since $z \in M$ and $a \in T$). So $zb^{-r}I \subseteq R$, and then $zb^{-r} \in I^{-1}$. Hence $b^{-r}M \subseteq I^{-1}$ and therefore $b^{-r}M = I^{-1}$. So $I_v = (R : I^{-1}) = (R : b^{-r}M) = b^r M^{-1}$. Since $II^{-1} = R$ (respectively $I_v = cR$), then $1 = \sum_{i=1}^{s} g_i h_i$, where $g_i \in I$ and $h_i \in I^{-1}$ (respectively $b^r M^{-1} = cR$). Now, for each $i \in \{1, \ldots, s\}$, write $g_i = \alpha_i(ab)^r + \beta_i b^r = b^r(\alpha_i a^r + \beta_i)$ and $h_i = b^{-r} f_i$, where $\alpha_i, \beta_i \in R$ and $f_i \in M$. Then $1 = \sum_{i=1}^{s} g_i h_i = \sum_{i=1}^{s} b^r(\alpha_i a^r + \beta_i)b^{-r} f_i = \sum_{i=1}^{s}(\alpha_i a^r + \beta_i) f_i \in M$, since $(\alpha_i a^r + \beta_i) \in T$ and $f_i \in M$, which is absurd (respectively $R \subset T \subseteq (M^{-1} : M^{-1}) = (b^r M^{-1} : b^r M^{-1}) = (cR : cR) = R$, which is absurd). It follows that $k \subseteq K$ is a root extension.

Now, we prove that if R is *AGCD*-domain, then M is a t-ideal of T. Assume that R is *AGCD*-domain and suppose that M is not a t-ideal of T. Since M is maximal in T, then $M_{t_1} = T$, where t_1 denotes the t-operation with respect to T. Then there exists a f.g. ideal $J = \sum_{i=1}^{r} a_i T$ of T such that $J \subseteq M$ and $(T : J) = T$. Hence $T = (T : M) = (M : M) = M^{-1}$. So $(R : M^2) = ((R : M) : M) = (T : M) = T$ and by induction on n, $(R : M^n) = T$. Therefore $(M^n)_v = M$ for each n. Set $I = \sum_{i=1}^{r} a_i R$. Then $I \subseteq M$ and $IT = J$. So $T = M^{-1} \subseteq I^{-1} \subseteq (T : IT) = (T : J) = T$. Hence $I_v = M$. Since R is *AGCD*-domain, then there exists a natural number s such that $(a_1^s, \ldots, a_r^s)_v$ is principal (by induction, or the remark following [5, Lemma 3.3]). Set $A = (a_1^s, \ldots, a_r^s)$ and $A_v = cR$. Since I is f.g., then there exists a natural number n such that $I^n \subseteq A \subseteq I^s$ (it suffices to take $n = s^s$). So $M = (M^n)_v = ((I_v)^n)_v = (I^n)_v \subseteq A_v = cR \subseteq (I^s)_v = ((I_v)^s)_v = (M^s)_v = M$. Hence $M = cR$ and therefore $T = (M : M) = (cR : cR) = R$, which is a contradiction. It follows that M is a t-ideal of T. □

Theorem 2.2. *For the diagram* (□), *R is an AV-domain* (*respectively AP-domain*) *if and only if T and D are AV-domains* (*respectively AP-domains*) *and the extension $k \subseteq K$ is a root extension.*

Proof. (⇒) By [5, Lemma 4.5], T is an *AV*-domain (respectively *AP*-domain) as on overring of R, and by [5, Theorem 4.10], $D = R/M$ is an *AV*-domain (respectively *AP*-domain). Also by Lemma 2.1, the extension $k \subseteq K$ is a root extension.

For the converse, we need the following lemmas.

Lemma 2.3. *For the diagram* (□*), *R is an AV-domain if and only if T and D are.*

Proof. Assume that T and D are *AV*-domains. By [5, Theorem 5.6], T is t-local (i.e., T is local with maximal ideal M and M is a t-ideal of T). Let $0 \neq x \in qf(R) = qf(T)$. Since T is an *AV*-domain, then there exists a nonzero natural number n such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in M$, then $x^n \in R$, as desired. Assume that $x^n \notin M$. Then $x^{-n} \in T$. Set $\lambda = \varphi(x^n)$. Then $0 \neq \lambda \in K = qf(D)$. Since D is an *AV*-domain, then there exists a natural number r such that $\lambda^r \in D$ or $\lambda^{-r} \in D$. If $\lambda^r \in D$, then $\varphi(x^{nr}) = \lambda^r \in D$. So $x^{nr} \in R$. If $\lambda^{-r} \in D$, then $\varphi(x^{-nr}) = \lambda^{-r} \in D$. So $x^{-nr} \in R$. It follows that R is an *AV*-domain.

The converse is clear. □

Lemma 2.4. *For the diagram* ($\square$), *assume that* $D = k$ *is a field. Then* R *is an AV-domain if and only if* T *is an AV-domain and* $k \subseteq K$ *is a root extension.*

Proof. ($\Rightarrow$) Follows from Lemma 2.1 and the fact that T is an overring of R.

($\Leftarrow$) Let $0 \neq x \in qf(R) = qf(T)$. Since T is an *AV*-domain, then there exists a nonzero natural number n such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in M$, then $x^n \in R$, as desired. Assume that $x^n \notin M$. Then $\lambda = \varphi(x^n) \in K \setminus \{0\}$. Since $k \subseteq K$ is a root extension, then there exists a natural number r such that $\lambda^r \in k$. Hence $\varphi(x^{nr}) = \lambda^r \in k$ and then $x^{nr} \in R$. It follows that R is an *AV*-domain. $\square$

Proof of Theorem 2.2 (*continued*). Consider the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M = K \end{array}$$

By Lemma 2.4, R_0 is an *AV*-domain and by Lemma 2.3, R is an *AV*-domain.

Now, assume that T and D are *AP*-domains and $k \subseteq K$ is a root extension. By [5, Theorem 5.8], it suffices to show that R_P is an *AV*-domain for each maximal ideal P of R. Let P be a maximal ideal of R. If $M \nsubseteq P$, then there is a unique maximal ideal Q of T such that $Q \cap R = P$ and $R_P = T_Q$. Since T is an *AP*-domain, then $R_P = T_Q$ is an *AV*-domain, as desired. Assume that $M \subseteq P$. Then there exists a unique maximal ideal p of D such that $P = \varphi^{-1}(p)$. In this case, consider the following diagram:

$$\begin{array}{ccc} R_P & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K = T/M \end{array}$$

Since T_M and D_p are *AV*-domains and $k \subseteq K$ is a root extension, by the *AV*-domain case, R_P is an *AV*-domain. It follows that R is an *AP*-domain. $\square$

Theorem 2.5. *For the diagram* ($\square$), *assume that* $D = k$ *is a field. Then* R *is AGCD-domain if and only if* T *is AGCD-domain,* M *is a* t*-ideal of* T *and* $k \subseteq K$ *is a root extension.*

Proof. ($\Rightarrow$) Assume that R is *AGCD*-domain. By Lemma 2.1, $k \subseteq K$ is a root extension and M is a t-ideal of T. So $R \subseteq T$ is a root extension and therefore $T \subseteq R'$. Hence $T' = R'$ and therefore T' is *AGCD*-domain by [5, Theorem 5.9(i)]. Since $R \subseteq R' = T'$ is a root extension [5, Theorem 5.9(ii)], then $T \subseteq T'$ is a root extension. Now, by

[5, Theorem 5.9] it suffices to show that T is t-linked under T'. Let $x_1, \dots, x_s \in T \setminus \{0\}$ such that $((x_1, \dots, x_s)T')_{v'_1} = T'$ where v'_1 is the v-operation with respect to T'. Since $R \subseteq T$ is a root extension, then for each $i \in 1, \dots, s$, there exists nonzero natural number n_i such that $x_i^{n_i} \in R$. Set $n = \prod_{i=1}^{s} n_i$. Then $x_i^n \in R$ for each $i \in 1, \dots, s$. Set $J = (x_1, \dots, x_s)T$. By hypothesis $(T' : JT') = T'$ (since $(JT')_{v'_1} = T'$). Then for each positive integer p, $(T' : J^pT') = (T' : (JT')^p) = T'$. Since J is a f.g. ideal of T, there exists a natural number r such that $J^r \subseteq (x_1^n, \dots, x_s^n)T \subseteq J^n$. So $J^rT' \subseteq (x_1^n, \dots, x_s^n)T' \subseteq J^nT'$. Hence $T' = (T' : J^nT') \subseteq (T' : (x_1^n, \dots, x_s^n)T') \subseteq (T' : J^rT') = T'$. Therefore $(T' : (x_1^n, \dots, x_s^n)T') = T'$. Since $R' = T'$, then $(R' : (x_1^n, \dots, x_s^n)R') = R'$. Since R is *AGCD*-domain, then $(R : (x_1^n, \dots, x_s^n)R) = R$ [5, Theorem 5.9(iii)]. Since T is t-linked over R then $(T : (x_1^n, \dots, x_s^n)T) = T$. Now, since $(x_1^n, \dots, x_s^n)T \subseteq J$, then $T \subseteq (T : J) \subseteq (T : (x_1^n, \dots, x_s^n)T) = T$, and therefore $(T : J) = T$, as desired. It follows that T is *AGCD*-domain.

($\Leftarrow$) Since $k \subseteq K$ is a root extension, then $R \subseteq T$ is a root extension and therefore $T' = R'$. Since T is *AGCD*-domain, then $R' = T'$ is *AGCD*-domain and $T \subseteq T' = R'$ is a root extension [5, Theorem 5.9]. Hence $R \subseteq R'$ is a root extension too. It suffices to show that R is t-linked under R'. Let $x_1, \dots, x_s \in R \setminus \{0\}$ such that $((x_1, \dots, x_s)R')_{v'} = R'$ where v' is the v-operation with respect to R'. Set $I = (x_1, \dots, x_s)R$. By hypothesis $(T' : IT') = (R' : IR') = R' = T'$. Since T is *AGCD*-domain, then $(T : IT) = T$. Since IT is a f.g. ideal of T, then $(IT)_{t_1} = (IT)_{v_1} = T$, where t_1 and v_1 are respectively the t- and v-operations with respect to T. Since M is a t-ideal of T, then $IT \nsubseteq M$. So $I \nsubseteq M$. Since M is a maximal ideal of R, then $I + M = R$. So there exists $a \in I$ and $m \in M$ such that $1 = a + m$. Now, for each $z \in I^{-1}$, $z = za + zm$. Since $I^{-1} \subseteq (T : IT) = T$, then $z \in T$. Since $m \in M$, then $zm \in M \subseteq R$. Also $za \in II^{-1} \subseteq R$. Then $z = za + zm \in R$. It follows that $I^{-1} = R$ and therefore $((x_1, \dots, x_s)R)_v = R$. Now, by [5, Theorem 5.9], R is *AGCD*-domain. □

Combining Theorems 2.2 and 2.5, we obtain the complete characterization of when the pullback of the diagram (□) is almost Bézout when $D = k$ is a field. We note that the assumption on M to be a t-ideal of T is not needed since it is well known that for an *AB*-domain, every prime ideal is a t-prime [5, Corollary 5.4].

Corollary 2.6 (cf. [2, Theorem 3.5]). *For the diagram (□), assume that $D = k$ is a field. Then R is an AB-domain if and only if T is an AB-domain, and $k \subseteq K$ is a root extension.*

Proof. Assume that R is an *AB*-domain. Then R is *AGCD*-domain. By Lemma 2.1, $k \subseteq K$ is a root extension. Since T is an overring of R, then T is an *AB*-domain. Conversely, since T is an *AB*-domain, then T is *AGCD*-domain and M is a t-ideal of T. By Theorem 2.5, R is *AGCD*-domain. Since T is an *AP*-domain, by Theorem 2.2, R is an *AP*-domain. Therefore R is an *AB*-domain. □

Theorem 2.7. *For the diagram (□*), assume that T is local with maximal ideal M. Then R is AGCD-domain if and only if T and D are AGCD-domains and M is a t-ideal of T.*

Proof. ($\Leftarrow$) Assume that T and D are *AGCD*-domains and M is a t-ideal of T. Then T is t-local and therefore T is an *AV*-domain [5, Theorem 5.6]. Let $0 \neq a, b \in R$.

Case 1. $a, b \in R \setminus M$. Set $\varphi(a) = \lambda$ and $\varphi(b) = \mu$. Then $0 \neq \lambda, \mu \in D$. Since D is *AGCD*-domain, then there exists a nonzero natural number n such $((\lambda^n, \mu^n)D)_{v_D}$ is principal in D, where v_D is the v-operation with respect to D. Set $((\lambda^n, \mu^n)D)_{v_D} = \alpha D$ for some nonzero $\alpha \in D$. Set $\alpha = \varphi(c)$ for some $c \in R \setminus M$, $J = (\lambda^n, \mu^n)D$ and $I = (a^n, b^n)R$. Clearly $\varphi^{-1}(J) = I + M$ and $\varphi^{-1}(\alpha D) = cR + M$. Since T is local with maximal ideal M, then each ideal of R is comparable to M. Since a, b and $c \in R \setminus M$, then $M \subset I$ and $M \subset cR$. Hence $\varphi^{-1}(J) = I + M = I$ and $\varphi^{-1}(\alpha D) = cR + M = cR$. Now, by [7, Proposition 1.8.(a(2))], $I_v = (\varphi^{-1}(J))_v = \varphi^{-1}(J_{v_D}) = \varphi^{-1}(\alpha D) = cR$, as desired.

Case 2. $a \in M$ and $b \notin M$ (respectively $a \notin M$ and $b \in M$). Then $b^{-1} \in T$ (respectively $a^{-1} \in T$). So $ab^{-1} \in M$ (respectively $ba^{-1} \in M$). Then $a \in bM \subseteq bR$ and therefore $(a, b)R = bR$ (respectively $b \in aM \subseteq aR$, so $(a, b)R = aR$), as desired.

Case 3. $a, b \in M$. Since T is an *AV*-domain, then there exists a nonzero natural number n such that $(a/b)^n \in T$, or $(b/a)^n \in T$. Assume, for example, that $(a/b)^n \in T$. If $(a/b)^n \in M$, then $a^n \in b^n M \subseteq b^n R$. Therefore $(a^n, b^n)R = b^n R$, as desired. So we may assume that $(a/b)^n \notin M$. Set $\lambda = \varphi((a/b)^n)$. Since $0 \neq \lambda \in K = qf(D)$, we can write $\lambda = \mu/\alpha$ for some $0 \neq \mu, \alpha \in D$. Also, write $\mu = \varphi(c)$ and $\alpha = \varphi(d)$ for some $c, d \in R \setminus M$. Then $\varphi((a/b)^n) = \lambda = \mu/\alpha = \varphi(c)/\varphi(d)$ implies that $d(a/b)^n - c \in M$. Since $c \notin M$, then $d(a/b)^n \in R \setminus M$. Now, we consider the two elements $d(a/b)^n$ and d of $R \setminus M$. By Case 1, there exists a nonzero natural number r such that $((d^r, (d(a/b)^n)^r)R)_v$ is principal in R. Set $((d^r, (d(a/b)^n)^r)R)_v = zR$. Then it is easy to see that $((a^{nr}, b^{nr})R)_v = zb^{nr}d^{-r}R$, as desired. It follows that R is *AGCD*-domain.

($\Rightarrow$) Assume that R is *AGCD*-domain. By Lemma 2.1, M is a t-ideal of T. Let $0 \neq a, b \in T$. If a or b is not in M, since T is local with maximal ideal M, then $a^{-1} \in T$ or $b^{-1} \in T$. Therefore $(a, b)T$ is principal in T, as desired. So we may assume that $a, b \in M$. Since R is *AGCD*-domain, then there exists a nonzero natural number n such that $((a^n, b^n)R)_v = cR$. Set $I = (a^n, b^n)R$. Since $T = R_M$ and I is a f.g. ideal of R, then $(T : IT) = (R_M : IR_M) = (R : I)R_M = c^{-1}R_M = c^{-1}T$. Hence $((a^n, b^n)T)_{v_1} = cT$, as desired. It follows that T is *AGCD*-domain. Now it's easy to see that D is *AGCD*-domain. $\square$

Combining Theorems 2.5 and 2.7, we obtain the complete characterization of when the pullback of the diagram ($\square$) is *AGCD*-domain in the local case.

Theorem 2.8. *For the diagram ($\square$), assume that T is local with maximal ideal M. Then R is AGCD-domain if and only if T and D are AGCD-domains, M is a t-ideal of T and $k \subseteq K$ is a root extension.*

Proof. ($\Leftarrow$) Consider the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M = K \end{array}$$

By Theorem 2.5, R_0 is *AGCD*-domain and it's well known that R_0 is local with maximal ideal M and M is a v-ideal of R_0. By Theorem 2.7, R is *AGCD*-domain.

($\Rightarrow$) Follows easily from Theorems 2.5 and 2.7. $\square$

Combining Theorems 2.2 and 2.8, we obtain the complete characterization of when the pullback of the diagram ($\square$) is an *AB*-domain in the local case.

Theorem 2.9. *For the diagram* ($\square$), *assume that* T *is local with maximal ideal* M. *Then* R *is an AB-domain if and only if* T *and* D *are AB-domains, and* $k \subseteq K$ *is a root extension.*

3. More about *AGCD*-domains

A Noetherian *GCD*-domain is well known to be a *UFD*. In [2, Theorem 3.9], an *AGCD*-domain analogue of this result was sought. We start this section by extending this result to the class of strong Mori domains (i.e., domains satisfying the *acc* on w-ideals) which contains properly the class of Noetherian domains. Recall that an integral domain R is said to be weakly factorial (respectively almost weakly factorial) if each nonzero nonunit element of R (respectively some power of each nonzero nonunit of R) is a product of primary elements. R is weakly factorial (respectively almost weakly factorial) if and only if $R = \bigcap\{R_P / \operatorname{ht} P = 1\}$, where this intersection has finite character and the t-class group $\mathrm{Cl}_t(R)$ is trivial [4] (respectively torsion [3]).

Theorem 3.1. *Let* R *be an SM domain. If* R *is AGCD-domain, then* R *is almost weakly factorial.*

Proof. Assume that R is an *SM* which is *AGCD*-domain. By [5, Theorem 3.4], $\mathrm{Cl}_t(R)$ is torsion. By [6, Theorem 1.9], $R = \bigcap\{R_M / M \in w\text{-Max}(R)\}$ and this intersection has finite character. It suffices to show that $\operatorname{ht} M = 1$ for each $M \in w\text{-Max}(R)$. Let M be a w-maximal ideal of R. By [1, Corollary 2.17], or [10, Lemma 2.1], $w\text{-Max}(R) = t\text{-Max}(R)$. Since R is a Mori domain, then R is a *TV*-domain (i.e., $t = v$ [9]). Hence $M = M_w = M_t = M_v$. Also by [10, Theorem 3.1], M is of finite type. Set $M = I_v$ for some f.g. subideal $I = (a_1, \ldots, a_s)R$ of M. Since R is *AGCD*-domain, then there exists a nonzero natural number n such that $((a_1^n, \ldots, a_s^n)R)_v$ is principal in R. Set $(a_1^n, \ldots, a_s^n)R = A$ and $A_v = bR$ for some nonzero $b \in R$. Now, we claim that M is minimal over bR. Indeed, clearly $bR = A_v \subseteq I_v = M$ and if Q is minimal over bR with

$Q \subseteq M$, then Q is t-prime (since any minimal prime ideal over a t-ideal is t-prime). Since $A \subseteq A_v = bR \subseteq Q$, then $I \subseteq Q$. Hence $M = I_v = I_t \subseteq Q_t = Q \subseteq M$. So $Q = M$ and therefore M is minimal over bR. Now, since R is an *SM* domain, then R satisfies the *PIT* (Principal Ideal Theorem) for w-ideals [6, Corollary 1.11, or Corollary 1.12]. Then $\operatorname{ht} M = 1$, as desired. It follows that R is almost weakly factorial. □

An *SM* domain which is *AGCD*-domain is not necessarily Noetherian. It suffices to consider $R = k[\{X_\alpha\}]$, where $\{X_\alpha\}$ is a set of infinite indeterminates over the field k.

Theorem 3.2. *Let R be a domain and I a nonzero ideal of R. If R is AGCD-domain, then $(I_v : I_v)$ is AGCD-domain.*

Proof. Set $M = II^{-1}$ and $T = (I_v : I_v)$. It is well known that $M^{-1} = (M : M) = (I_v : I_v)$. Let $0 \neq x, y \in (I_v : I_v) = M^{-1}$ and let $0 \neq a \in I_v$. Then ax, and ay are nonzero elements of R. Since R is *AGCD*-domain, then there exists a nonzero natural number n such that $(((ax)^n, (ay)^n)R)_v = cR$ for some nonzero c of R. Set $J = ((ax)^n, (ay)^n)R$. Then $(T : JT) = (M^{-1} : JM^{-1}) = ((R : M) : JM^{-1}) = (R : JMM^{-1}) = (R : JM) = ((R : J) : M) = ((R : J_v) : M) = (c^{-1}R : M) = c^{-1}M^{-1} = c^{-1}T$. Hence $(a^n(x^n, y^n)T)_{v_1} = (((ax)^n, (ay)^n)T)_{v_1} = (JT)_{v_1} = cT$. Then $((x^n, y^n)T)_{v_1} = ca^{-n}T$, as desired. It follows that $T = (I_v : I_v)$ is *AGCD*-domain. □

Proposition 3.3. *Let R be an integral domain and X an indeterminate over R. Then $R[X]$* (*respectively $R[[X]]$*) *is an AP-domain if and only if R is a field.*

Proof. Suppose that R is an *AP*-domain. Let $0 \neq a \in R$. Then consider the elements a and X in R. There exists a nonzero natural number n such that $J = (a^n, X^n)R[X]$ is invertible in $R[X]$. On the other hand it is easy to see that $J^{-1} = R[X]$. Hence $R[X] = JJ^{-1} = J$. Then a is a unit in R and therefore R is a field.

For $R[[X]]$, it suffices to substitute $R[[X]]$ to $R[X]$. □

Corollary 3.4. *Let $R = A + XB[[X]]$* (*respectively $R = A + XB[X]$*). *Then R is an AB-domain* (*respectively AP-domain*) *if and only if A is an AB-domain* (*respectively AP-domain*), *$B = K$ is a field and $qf(A) = k \subseteq K$ is a root extension.*

Proof. Since $B[[X]]$ (respectively $B[X]$) is an overring of R, then it's an *AB*-domain (respectively *AP*-domain). By Proposition 3.3, $B = K$ is a field. Now, we conclude using Theorems 2.2, 2.5 and 2.9, by considering the following diagram:

$$\begin{array}{ccc} R = A + XK[[X]] & \longrightarrow & A \\ \downarrow & & \downarrow \\ T = K[[X]] & \longrightarrow & K[[X]]/XK[[X]] = K \end{array}$$

□

Acknowledgment

I express my sincere thanks to the referee for his/her helpful suggestions.

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Further reading

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