Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Pure and Applied Algebra 200 (2005) 37–50
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Semistar-operations of finite character on integral domains

Abdeslam Mimouni[1]

Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

Received 5 October 2002; received in revised form 18 November 2004
Available online 5 February 2005
Communicated by E.M. Friedlander

Abstract

In this paper, we investigate the semistar-operations of finite character on integral domains. We state a conditions under which the semistar-operation defined by a family of overrings of a domain R is of finite character. This notion leads us to give a new characterization of Prüfer domains and characterize Prüfer and Noetherian domains R for which each semistar-operation is of finite character. It turns out that R must be conducive (so local and one-dimensional) in the Noetherian case and conducive and each overring of R is divisorial for the Prüfer case. We also show that $3 + dim\ R \leqslant |SFc(R)|$ for each nonlocal domain R and we characterize domains for which the equality holds.

MSC: 13G05; 13A15; 13F05

1. Introduction

In 1994, Okabe and Matsuda [18] introduced the notion of semistar-operations. This concept extends the classical concept of star-operations, as developed in Gilmer's book [7], and hence the related classical theory of ideal systems based on the work of W. Krull, E. Noether, H. Prüfer, and P. Lorenzen. Since then, many investigations of semistar-operations have been done (for instance see [5,12–18]).

[1] Partially supported by KFUPM.

E-mail address: amimouni@kfupm.edu.sa (A. Mimouni).

0022-4049/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2004.12.019

Let R be an integral domain with quotient field K, $\bar{F}(R)$ the set of all nonzero R-submodules of K, $F(R)$ the set of all nonzero fractional ideals of R, i.e., all $A \in \bar{F}(R)$ such that $dA \subseteq R$ for some nonzero $d \in R$, and $f(R)$ the set of all nonzero finitely generated R-submodules of K. Then $f(R) \subseteq F(R) \subseteq \bar{F}(R)$.

A mapping $\bar{F}(R) \longrightarrow \bar{F}(R)$, $E \longrightarrow E^*$ is called a semistar-operation on R if for all $x \in K$ and $E, F \in \bar{F}(R)$:

(1) $(xE)^* = xE^*$.
(2) $E \subseteq E^*$ and $E \subseteq F \Longrightarrow E^* \subseteq F^*$.
(3) $E^{**} = E^*$.

If $E \in \bar{F}(R)$, then $E^* \in \bar{F}(R^*) \subseteq \bar{F}(R)$. The R-submodules of K belonging to $\bar{F}^*(R) := \{E^*/E \in \bar{F}(R)\}$ are called semistar R-submodules of K. Similarly, we can consider $F^*(R) := \{E^*/E \in F(R)\}$ and $f^*(R) := \{F^*/F \in f(R)\}$. It is easy to see that $F^*(R) \subseteq F(R^*)$, but in general $F(R^*) \nsubseteq F(R)$. Also $\bar{F}^*(R) \subseteq \bar{F}(R^*)$ and this inclusion may be strict, see [5, Remark 1.0.(b)].

A semistar-operation $*$ on R is proper if $R \subset R^*$. However, if $R = R^*$, then * restricted to $F(R)$ defines a star-operation on R.

The map $E \longrightarrow E^e := K$, for each $E \in \bar{F}(R)$ defines a semistar-operation on R called the e-operation and the map $E \longrightarrow E^{\bar{d}} := E$ defines a trivial semistar-operation called the $\bar{d}$-operation.

It is easy to see that each star-operation $*$ on R can be extended to a semistar-operation $\bar{*}$ as follows: $E^{\bar{*}} = E^*$ if $E \in F(R)$ and $E^{\bar{*}} = K$ if $E \in \bar{F}(R) \backslash F(R)$. The extension of the v-(respectively t-) operation will denoted by $\bar{v}$ (respectively $\bar{t}$).

A semistar-operation $*$ on R is called of finite character (or of finite type) if $E^* = \cup\{F^*/F \in f(R), F \subseteq E\}$ for each $E \in \bar{F}(R)$. For each semistar-operation $*$ on R, we associate a semistar-operation of finite character $*_f$ defined by $E^{*_f} = \cup\{F^*/F \in f(R), F \subseteq E\}$ for each $E \in \bar{F}(R)$. Obviously, a semistar-operation $*$ is of finite character if and only if $* = *_f$. Note that $\bar{v}_f = \bar{t}$.

Let $\mathscr{R} = \{(R_\alpha, *_\alpha)\}_{\alpha\in\Omega}$ be a family of overrings of R, where $*_\alpha$ is a semistar-operation on R_α. Then the map $E \longrightarrow E^* = \bigcap_{\alpha\in\Omega}(ER_\alpha)^{*_\alpha}$ is a semistar-operation on R called the semistar-operation defined by the family $\mathscr{R}$, and will be denoted by $*_{\mathscr{R}}$. In particular, if S is an overring of R and the family $\mathscr{R}$ is reduced to $(S, \bar{d})$, we write $*_{\{S\}}$ instead $*_{\mathscr{R}}$. Clearly $e = *_{\{K\}}$ and $\bar{d} = *_{\{R\}}$.

Let $S(R)$ denote the set of all semistar-operations on R, $SFc(R)$ the set of all semistar-operations of finite character on R, $S'(R)$ the set of all star-operations on R, $[R, K]$ the set of all overrings of R, $\mathrm{Spec}(R)$ the set of all prime ideals of R and for a set X, let $|X|$ denote the cardinality of X.

The purpose of the present paper is to study the semistar-operations of finite character on integral domains. In the first part, we give a partial answer to a problem cited by Fontana and Huckaba [5], and listed in a list of one hundred open problems by Chapman and Glaz (see [3, Problem 44]) by stating conditions under which the semistar-operation defined by a family $\mathscr{R} = \{(R_\alpha, *_\alpha)\}_{\alpha\in\Omega}$ of overrings of R is of finite character in the context of conducive domains. Precisely, we prove that *if* $R^{*_{\mathscr{R}}} = \bigcap_{\alpha\in\Omega}(R_\alpha)^{*_\alpha}$ *is locally finite and each* $*_\alpha$ *is of finite character*, *then* $*_{\mathscr{R}}$ *is of finite character*. We also prove, without the "conductivity" assumption on R, that *if Ω is finite and each $*_\alpha$ is of finite character, then so is $*_{\mathscr{R}}$*. It turns out that $*_{\{S\}}$ is of finite character for each overring S of R.

The second section is devoted to the study of the domains R for which $\bar{F}^*(R) = \bar{F}(R^*)$ (respectively, $F^*(R) = F(R^*)$) for each semistar-operation of finite character $*$ on R in the context of integrally closed domains. This leads us to give a new characterization of Prüfer domains, that is, *Let R be an integrally closed domain. Then R is Prüfer if and only if $\bar{F}^*(R) = \bar{F}(R^*)$ (respectively, $F^*(R) = F(R^*)$) for each $* \in SFc(R)$ if and only if $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$*. We also characterize domains for which each semistar-operation is of finite character, that is $S(R)=SFc(R)$, in the context of Noetherian and Prüfer domains. It turns out that such domains must be conducive, so local and one-dimensional in Noetherian case and conducive and each overring of R is divisorial in the Prüfer case.

The last section deals with a discussion about the cardinality of $SFc(R)$. In [17], it was shown that $1 + \dim R \leqslant |SFc(R)|$ and the equality holds if and only if R is a valuation domain. By virtue of this result, we focus our attention to the case where $2 + \dim R \leqslant |SFc(R)|$. We firstly prove the following theorem: *Let R be a nonlocal domain. Then $3 + \dim R \leqslant |SFc(R)|$ and the equality holds if and only if R is a Prüfer domain with Spec(R) reduced to a unique Y-graph*. Furthermore, we characterize domains for which $|SFc(R)| = 2 + \dim R$.

2. Finiteness conditions of semistar-operations defined by a family of overrings

Before stating our first result, we recall that a domain R is said to be conducive if $(R : T) \neq (0)$ for each overring $T \subset K$ of R (see [4]). Conducive domains have particular interest in the study of the semistar-operations and link this notion to the notion of star-operations. Indeed, it is easy to see that for such domains R, $\bar{F}(R) = F(R) \cup \{K\}$ and each star-operation on R has a unique extension to a semistar-operation on R. The following Proposition characterizes conducive domains in terms of semistar-operations.

Proposition 2.1. *Let R be a domain. The following are equivalent*:

(i) *R is a conducive domain*;
(ii) *For each overring $T \subset K$ of R, and for each $* \in S(R)\backslash\{e\}$, $T^* \subset K$*;
(iii) *For each valuation overring $V \subset K$ of R, and for each $* \in S(R)\backslash\{e\}$, $V^* \subset K$*;
(iv) *There is a valuation overring $V \subset K$ of R such that $V^* \subset K$ for each $* \in S(R)\backslash\{e\}$.*

Proof. (i) $\Longrightarrow$ (ii). Let T be an overring of R with $T \subset K$ and let $* \in S(R)\backslash\{e\}$. By (i), there is $0 \neq d \in R$ such that $dT \subseteq R$. So $dT^* = (dT)^* \subseteq R^*$. If $T^* = K$, then $K = dK = dT^* \subseteq R^* \subseteq K$. So $R^* = K$ and therefore $* = e$, which is absurd. Hence $T^* \subset K$.
(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) Trivials.
(iv) $\Longrightarrow$ (i) Assume that there is a valuation overring $V \subset K$ of R such that $V^* \subset K$ for each $* \in S(R)\backslash\{e\}$. Suppose that $(R : V) = 0$. Let $*$ be the semistar-operation on R defined by $A^* = A$ if $A \in F(R)$ and $A^* = K$ if $A \in \bar{F}(R)\backslash F(R)$. Since $R^* = R$, then $* \neq e$. Since $(R : V) = 0$, then $V^* = K$. This yields to a contradiction with (iv). So $(R : V) \neq 0$ and therefore R is conducive, [4, Theorem 3.2]. □

Proposition 2.2. *Let R be a domain. Then* $\bar{v} = \bar{d}$ *if and only if R is a conducive domain which is divisorial.*

Proof. Clearly $\bar{v}=\bar{d}$ implies that $v=d$ and therefore R is divisorial. Now, if T is an overring of R such that $(R : T) = 0$, then $K = T^{\bar{v}} = T^{\bar{d}} = T$. Hence R is conducive. Conversely, R divisorial implies that $v = d$ and R conducive implies that $\bar{F}(R) = F(R) \cup \{K\}$. Since $K = K^{\bar{v}} = K^{\bar{d}}$, then $\bar{v} = \bar{d}$. □

A star-operation on nonconducive domain R can have more than one extension to a semistar-operation. Indeed, let R be a nonconducive *PID* (for example $R = Z$ the ring of integers or $R = k[X]$, where k is a field and X an indeterminate over k). Clearly $\bar{v}$ and $\bar{d}$ are two (different) extensions of d.

The following two theorems state conditions under which the semistar-operation defined by a family of overrings of a domain R is of finite character.

Theorem 2.3. *Let R be a conducive domain,* $\mathscr{R}=\{(R_\alpha, *_\alpha)\}_{\alpha\in\Omega}$ *a family of overrings of R, where* $*_\alpha$ *is a semistar-operation of finite character on* R_α *and* $*_{\mathscr{R}}$ *be the semistar-operation on R defined by* $\mathscr{R}$. *If* $R^{*_{\mathscr{R}}} = \bigcap (R_\alpha)^{*_\alpha}$ *is locally finite, then* $*_{\mathscr{R}}$ *is of finite character.*

Proof. Let $(*_{\mathscr{R}})_f$ denote the semistar-operation of finite character associated to $*_{\mathscr{R}}$. Our aim is to prove that $*_{\mathscr{R}} = (*_{\mathscr{R}})_f$. Since R is conducive, then $\bar{F}(R) = F(R) \cup \{K\}$. Since $K^{(*_{\mathscr{R}})_f} = K = K^{*_{\mathscr{R}}}$, then it suffices to show that $A^{*_{\mathscr{R}}} = A^{(*_{\mathscr{R}})_f} = \bigcup\{J^{*_{\mathscr{R}}}/J \in f(R), J \subseteq A\}$ for each $A \in F(R)$. Let $A \in F(R)$, $0 \neq a \in A$ and $0 \neq d \in R$ such that $dA \subseteq R$. Then $da \in R \subseteq R^{*_{\mathscr{R}}} = \bigcap (R_\alpha)^{*_\alpha}$ which is locally finite, then there is $\alpha_1, \ldots, \alpha_n \in \Omega$ such that $(da)^{-1} \in (R_\alpha)^{*_\alpha}$ for each $\alpha \in \Omega\backslash\{\alpha_1, \ldots, \alpha_n\}$. Now, let $x \in A^{*_{\mathscr{R}}} = \bigcap_{\alpha\in\Omega}(AR_\alpha)^{*_\alpha}$. Then for each $i \in \{1, \ldots, n\}$, $x \in (AR_{\alpha_i})^{*_{\alpha_i}} = \bigcup\{J^{*_{\alpha_i}}/J \in f(R_{\alpha_i}), J \subseteq AR_{\alpha_i}\}$ since $*_{\alpha_i}$ is of finite character. So there is $J_i \in f(R_{\alpha_i})$ such that $J_i \subseteq AR_{\alpha_i}$ and $x \in (J_i)^{*_{\alpha_i}}$. Set $J_i = \sum_{j=1}^{j=r_i} b_{ij}R_{\alpha_i}$. Since $J_i \subseteq AR_{\alpha_i}$, then $b_{ij} \in AR_{\alpha_i}$ for each $j \in \{1, \ldots, r_i\}$. Set $b_{ij} = \sum_{t=1}^{t=n_{ij}} a_{ij,t}x_t$ where $a_{ij,t} \in A$ and $x_t \in R_{\alpha_i}$. Now, let L_i be the f.g. (fractional) ideal of R generated by all $a_{ij,t}$, $1 \leqslant j \leqslant r_i$, $1 \leqslant t \leqslant n_{ij}$. Then $L_i \subseteq A$ and $J_i \subseteq L_iR_{\alpha_i}$. So $x \in (J_i)^{*_{\alpha_i}} \subseteq (L_iR_{\alpha_i})^{*_{\alpha_i}}$. Let J be the f.g. ideal of R given by $J = aR + \sum_{i=1}^{i=n} L_i$. Then $J \subseteq A$ and for each $i \in \{1, \ldots, n\}$, $x \in (J_i)^{*_{\alpha_i}} \subseteq (L_iR_{\alpha_i})^{*_{\alpha_i}} \subseteq (JR_{\alpha_i})^{*_{\alpha_i}}$. Now, let $\alpha \in \Omega$. If $\alpha=\alpha_i$ for some $i \in \{1, \ldots, n\}$, then $x \in (JR_{\alpha_i})^{*_{\alpha_i}}$. Assume that $\alpha \in \Omega\backslash\{\alpha_1, \ldots, \alpha_n\}$. Then $(da)^{-1} \in (R_\alpha)^{*_\alpha}$. Since $dA \subseteq R$, then $dA^{*_{\mathscr{R}}} \subseteq R^{*_{\mathscr{R}}} \subseteq (R_\alpha)^{*_\alpha}$. So $xd \in (R_\alpha)^{*_\alpha}$. Hence $(da)^{-1}(dx) \in (R_\alpha)^{*_\alpha}$. Therefore $x = (da)^{-1}(dx)a \in J(R_\alpha)^{*_\alpha} \subseteq (J(R_\alpha)^{*_\alpha})^{*_\alpha} = (JR_\alpha)^{*_\alpha}$. It follows that $x \in \bigcap (JR_\alpha)^{*_\alpha} = J^{*_{\mathscr{R}}} \subseteq A^{(*_{\mathscr{R}})_f}$. Hence $A^{*_{\mathscr{R}}} \subseteq A^{(*_{\mathscr{R}})_f}$ and therefore $A^{*_{\mathscr{R}}} = A^{(*_{\mathscr{R}})_f}$. Hence $*_{\mathscr{R}}$ is of finite character. □

In case of finite family of overrings of R, the hypothesis of "locally finite" is always satisfies. However, the "conductivity" assumption on R is not needed as it shown by the following theorem.

Theorem 2.4. *Let R be a domain,* $\mathscr{R} = \{(R_i, *_i)\}_{1\leqslant i\leqslant n}$ *a finite family of overrings of R, where* $*_i$ *is a semistar-operation of finite character on* R_i. *Then the semistar-operation* $*_{\mathscr{R}}$ *is of finite character.*

Proof. Let $A \in \bar{F}(R)$ and let $x \in A^{*_{\mathscr{R}}} = \bigcap (AR_i)^{*_i}$. Then for each $i \in \{1, \dots, n\}$, $x \in (AR_i)^{*_i} = \bigcup \{J^{*_i}/J \in f(R_i), J \subseteq AR_i\}$. So there is $J_i \in f(R_i)$, $J \subseteq AR_i$ such that $x \in (J_i)^{*_i}$. As in the proof of Theorem 2.3, there is $L_i \in f(R)$ such that $L_i \subseteq A$ and $J_i \subseteq L_i R_i$. So $x \in (L_i R_i)^{*_i}$. Now, set $L = \sum_{i=1}^{i=n} L_i$. Then $L \in f(R)$, $L \subseteq A$ and for each $i \in \{1, \dots, n\}$, $x \in (J_i)^{*_i} \subseteq (L_i R_i)^{*_i} \subseteq (LR_i)^{*_i}$. Hence $x \in \bigcap (LR_i)^{*_i} = L^{*_{\mathscr{R}}}$ and therefore $A^{*_{\mathscr{R}}} = \bigcup \{J^{*_{\mathscr{R}}}/J \in f(R), J \subseteq A\}$.It follows that $*_{\mathscr{R}}$ is of finite character on R. □

Corollary 2.5. *Let R be a domain, T an overring of R and $*_\alpha$ a semistar-operation of finite character on T. Then the semistar-operation on R defined by $A^* = (AT)^{*_\alpha}$ is of finite character. In particular, the semistar-operation $*_{\{T\}}$ defined by T is a semistar-operation on R of finite character.*

3. The integrally closed case

In this section, we investigate semistar-operations in the context of integrally closed domains. Our aim is to give a new characterization of Prüfer domains via semistar-operations. Before stating our next result, we recall that, according to Zafrullah [19], a domain R is said to be an *fgv* domain if each finitely generated ideal is divisorial. Clearly R is an *fgv* domain if and only if the t-operation on R is trivial, that is $t = d$. Since $d \leqslant w \leqslant t$, where w is the w-operation, an *fgv* domain is a *TW* domain. We also recall that a domain R is Prüfer if and only if R is an *fgv* domain which is integrally closed.

Theorem 3.1. *Let R be an integrally closed domain which not local. The following statements are equivalent*:

(i) $\bar{F}^*(R) = \bar{F}(R^*)$ *for each* $* \in SFc(R)$.
(ii) $\bar{F}^*(R) = \bar{F}(R^*)$ *for each* $* \in SFc(R)$ *with* $R \subset R^*$.
(iii) $F^*(R) = F(R^*)$ *for each* $* \in SFc(R)$.
(iv) $F^*(R) = F(R^*)$ *for each* $* \in SFc(R)$ *with* $R \subset R^*$.
(v) *Each semistar-operation of finite character on R is defined by an overring of R, that is* $* = *_{\{R^*\}}$, *for each* $* \in SFc(R)$, *and hence* $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$.
(vi) *R is a Prüfer domain.*

Proof. (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are trivials.

Simultaneously, we prove (ii) $\Longrightarrow$ (vi) and (iv) $\Longrightarrow$ (vi). Let M be a maximal ideal of R. Consider the semistar-operation $*$ on R given by $A^* = (AR_M)^{\bar{t}_M}$, where $\bar{t}_M$ is the $\bar{t}$-semistar operation on R_M. By Corollary 2.5, $*$ is of finite character on R and $R^* = R_M \supset R$. By (ii) (resp. (iv)), $\bar{F}^*(R) = \bar{F}(R^*) = \bar{F}(R_M)$ (resp. $F^*(R) = F(R^*) = F(R_M)$). So, for each $A \in F(R_M)$, there is $B \in \bar{F}(R)$ (resp. $F(R)$) such that $A = B^* = (BR_M)^{\bar{t}_M}$. Then $A^{t_M} = A^{\bar{t}_M} = A^* = (B^*)^* = B^* = A$. Hence the t-operation on R_M is trivial. Since R_M is integrally closed, then R_M is a Prüfer domain and therefore a valuation domain. It follows that R is a Prüfer domain.

(vi) $\Longrightarrow$ (i) Assume that R is Prüfer. Let $* \in SFc(R)$ and let $A \in \bar{F}(R^*)$. Then $A^* = \bigcup \{J^*/J \in f(R), J \subseteq A\}$. Now, for each $J \in f(R)$ with $J \subseteq A$, $JJ^{-1} = R$ since R is Prüfer. So $1 = \sum_{i=1}^{i=r} a_i x_i$ where $a_i \in J$ and $x_i \in J^{-1}$ for each $i \in \{1, \dots, r\}$. Then, for each $z \in J^*$, $z = \sum_{i=1}^{i=r} a_i z x_i$. Since $J^{-1} = (R : J) \subseteq (R^* : J^*)$, then $zx_i \in R^*$ for each $i \in \{1, \ldots, r\}$. So $z = \sum_{i=1}^{i=r} a_i z x_i \in \sum_{i=1}^{i=r} a_i R^* \subseteq JR^* \subseteq AR^* \subseteq A$, since A is an R^*-module. Hence $J^* \subseteq A$ and therefore $A^* = A$. Since $\bar{F}(R^*) \subseteq \bar{F}(R)$ and $A^* = A$, then $A \in \bar{F}^*(R)$. Hence $\bar{F}(R^*) \subseteq \bar{F}^*(R)$, and therefore $\bar{F}(R^*) = \bar{F}^*(R)$.

(vi) $\Longrightarrow$ (iii) Similar to (vi) $\Longrightarrow$ (i) by replacing $\bar{F}(R^*)$ by $F(R^*)$. The fact that R is Prüfer forces that $J^* = JR^*$ for each $J \in f(R)$. So, for each $A \in F(R^*)$, $A = A^*$ and therefore $F^*(R) = F(R^*)$, as desired.

(v) $\Longrightarrow$ (vi) Let $\bar{t}$ be the $\bar{t}$-semistar operation on R. Since $\bar{t}$ is of finite character, by (v), $\bar{t}$ is defined by an overring of R. So $\bar{t} = *_{\{T\}}$ for some overring T of R. Since $R = R^{\bar{t}} = R^{*_{\{T\}}} = T$, then $\bar{t} = *_{\{T\}} = *_{\{R\}} = \bar{d}$. So the t-operation on R is trivial. Since R is integrally closed, then R is a Prüfer domain.

(vi) $\Longrightarrow$ (v) By virtue of Corollary 2.5, $\{*_{\{T\}}/T \in [R, K]\} \subseteq SFc(R)$. Let $* \in SFc(R)$ and set $T = R^*$. Our aim is to show that $* = *_{\{T\}}$. Let $A \in \bar{F}(R)$. Then $A^* = \bigcup\{J^*/J \in f(R), J \subseteq A\}$. Now, for each $J \in f(R)$ with $J \subseteq A$, $JJ^{-1} = R$ since R is Prüfer. Write $1 = \sum_{i=1}^{i=r} a_i x_i$ where $a_i \in A$ and $x_i \in J^{-1}$ for each $i \in \{1, \ldots, r\}$. Then, for each $z \in J^*$, $z = \sum_{i=1}^{i=r} a_i z x_i$. Since $J^{-1} = (R : J) \subseteq (R^* : J^*)$, then $zx_i \in R^*$ for each $i \in \{1, \ldots, r\}$. So $z = \sum_{i=1}^{i=r} a_i z x_i \in \sum_{i=1}^{i=r} a_i R^* \subseteq JR^* \subseteq AR^*$. Hence $J^* \subseteq JR^* \subseteq AR^*$ and therefore $A^* \subseteq AR^*$. Conversely, $AR^* \subseteq (AR^*)^* = (AR)^* = A^*$. Hence $A^* = AR^* = AT = A^{*_{\{T\}}}$ and therefore $* = *_{\{T\}}$. It follows that $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$. □

Corollary 3.2. *Let R be a nonlocal Krull domain. The following statements are equivalent*:

(i) $F^*(R) = F(R^*)$ *for each* $* \in S(R)$.
(ii) $F^*(R) = F(R^*)$ *for each* $* \in S(R)$ *with* $R \subset R^*$.
(iii) $F^*(R) = F(R^*)$ *for each* $* \in SFc(R)$.
(iv) $F^*(R) = F(R^*)$ *for each* $* \in SFc(R)$ *with* $R \subset R^*$.
(v) *R is a Dedekind domain*.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (iv) are trivials.

(iv) $\Longrightarrow$ (v) By Theorem 3.1, R is a Prüfer domain. Since R is Krull then R is a Dedekind domain.

(v) $\Longrightarrow$ (i) Let $* \in S(R)$ and let $A \in F(R^*)$. Since R^* is a Dedekind domain, then $A(R^* : A) = R^*$. Write $1 = \sum_{i=1}^{i=r} a_i x_i$ where $a_i \in A$ and $x_i \in (R^* : A)$ for each $i \in \{1, \ldots, r\}$. Set $B = \sum_{i=1}^{i=r} a_i R$. For each $z \in A^*$, $z = \sum_{i=1}^{i=r} a_i z x_i \in \sum_{i=1}^{i=r} a_i R^* = BR^*$ since $(R^* : A) = (R^* : A^*)$. So $A^* \subseteq BR^* \subseteq AR^* \subseteq A \subseteq A^*$. Then $A = A^* = BR^* \subseteq (BR^*)^* = B^* \subseteq A^*$. Hence $B^* = A^* = A$ and therefore $A \in F^*(R)$. So $F(R^*) \subseteq F^*(R)$, as desired.

(v) $\Longrightarrow$ (iii) Follows from Theorem 3.1. □

Our next result treats the integrally closed local case.

Proposition 3.3. *Let R be an integrally closed local domain. The following conditions are equivalent*.

(i) $\bar{F}^*(R) = \bar{F}(R^*)$ *for each* $* \in S(R)$ (*resp*. $* \in SFc(R)$).
(ii) $F^*(R) = F(R^*)$ *for each* $* \in S(R)$ (*resp*. $* \in SFc(R)$).

(iii) *R is a strongly discrete valuation (resp. valuation) domain.*

Moreover, with respect to $SFc(R)$, the three assertions are equivalent to:

(iv) $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$.

Proof. (i) $\Longrightarrow$ (iii) Since $\bar{F}^{\bar{v}}(R) = \bar{F}(R^{\bar{v}}) = \bar{F}(R)$ (resp. $\bar{F}^{\bar{t}}(R) = \bar{F}(R^{\bar{t}}) = \bar{F}(R)$), then R is divisorial (resp. an *fgv* domain). Since R is integrally closed, then R is Prüfer and therefore a valuation domain. Now, for the first part of (i), suppose that R is not strongly discrete. Then R has a nonzero idempotent prime P. Let $*$ be the semistar-operation on R defined by $A^* = (AR_P)^{\bar{v}_P}$, where $\bar{v}_P$ is the $\bar{v}$-semistar operation on R_P. By (i), $\bar{F}^*(R) = \bar{F}(R^*) = \bar{F}(R_P)$ implies that R_P is divisorial. By [8, Lemma 5.2], PR_P is principal, which is absurd since $P = P^2$. It follows that R is strongly discrete.

(ii) $\Longrightarrow$ (iii) Similar to (i) $\Longrightarrow$ (iii) by substituting $F(R)$ to $\bar{F}(R)$.

Now, we restrict to $SFc(R)$ and we will show the equivalence (iii) $\Longleftrightarrow$ (iv).

(iii) $\Longrightarrow$ (iv). Assume that R is a valuation domain. Let $* \in SFc(R)$ and set $T = R^*$. For each $J \in f(R)$, $J = aR$ for some nonzero $a \in J$. Then $J^* = (aR)^* = aR^* = aT = JT = J^{*\{T\}}$. Since $*$ is of finite character and $*|_{f(R)} = *_{\{T\}}|_{f(R)}$, then $* = *_{\{T\}}$. Now, by Corollary 2.5, $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$.

(iv) $\Longrightarrow$ (iii). Since $\bar{t} \in SFc(R)$, by (iii), there is $T \in [R, K]$ such that $\bar{t} = *_{\{T\}}$. So $R = R^{\bar{t}} = R^{*\{T\}} = T$. Then $\bar{t} = *_{\{T\}} = *_{\{R\}} = \bar{d}$ and therefore R is an *fgv* domain. Since R is integrally closed, then R is a Prüfer domain. Since R is local, then R is a valuation domain.

(iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii). The first part (with respect to $S(R)$) follows from [17, Corollary 14] and the fact that R is a conducive domain. The second part (with respect to $SFc(R)$) follows from (iii) $\Longleftrightarrow$ (iv) since for each $* \in SFc(R)$ and $A \in F(R^*)$, $A^* = AR^* = A$ and R is conducive. □

Corollary 3.4. *Let R be an integrally closed domain. Then $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$ if and only if R is a Prüfer domain.*

Proof. Follows immediately from Theorem 3.1 (if R is nonlocal) and Proposition 2.3 (if R is local). □

The next theorem characterizes Noetherian domains R such that $F^*(R) = F(R^*)$ for each $* \in SFc(R)$ with $R \subset R^*$.

Theorem 3.5. *Let R be a nonlocal Noetherian domain. The following conditions are equivalent*:

(i) *$F^*(R) = F(R^*)$ for each $* \in SFc(R)$ with $R \subset R^*$.*

(ii) *$dim\ R = 1$ and each proper overring of R is divisorial.*

Proof. (i) $\Longrightarrow$ (ii) Let M be a maximal ideal of R and consider the semistar-operation $* : \bar{F}(R) \longrightarrow \bar{F}(R)$ defined by $A^* = (AR_M)^{\bar{t}_M}$, where $\bar{t}_M$ is the $\bar{t}$-semistar operation on R_M. By Corollary 1.5, $*$ is of finite character and $R \subset R_M = R^*$. By (i), $F^*(R) = F(R^*) = F(R_M)$ and therefore the t-operation on R_M is trivial. Since R_M is Noetherian, then R_M is a *TV*-domain, and therefore R_M is divisorial. By [11, Theorem 222], $ht\,M = dim\ R_M = 1$. It follows that $dim\ R = 1$. Now, let T be a proper overring of R and let $*$ be the semistar-operation on R defined by $A^* = (AT)^{\bar{t}_T}$, where $\bar{t}_T$ is the $\bar{t}$-semistar-operation on T. By Corollary 2.5, $*$ is of finite character and $R \subset T = R^*$. By (i), $F^*(R) = F(R^*) = F(T)$. So the t-operation on T is trivial. Since $dim\ R = 1$, by [11, Theorem 93], T is Noetherian and $dim\ T = 1$. Hence T is divisorial.

(ii) $\Longrightarrow$ (i). Let $*$ be a semistar-operation on R of finite character with $R \subset R^*$ and set $T = R^*$. By (ii), T is a Noetherian divisorial domain. Let $A \in F(R^*) = F(T)$. Write $A = \sum_{i=1}^{i=n} a_i T$ for some nonzero $a_i \in A$. Let $B = \sum_{i=1}^{i=n} a_i R$. Then $B \in f(R) \subseteq F(R)$ and $A = BT$. So $A^* = (BT)^* = (BR^*)^* = (BR)^* = B^*$. On the other hand, since $T^* = (R^*)^* = R^* = T$, then the restriction $*|_{F(T)}$ of $*$ to $F(T)$ is a star-operation. Since T is divisorial, then $*|_{F(T)} = d$. Hence $A = A^* = B^*$ and therefore $A \in F^*(R)$. Hence $F(R^*) \subseteq F^*(R)$ and therefore $F(R^*) = F^*(R)$. $\square$

The next Theorem characterizes domains for which each semistar-operation is defined by an overring.

Theorem 3.6. *Let R be an integral domain. Then each semistar-operation on R is defined by an overring of R, that is $* = *_{\{R^*\}}$ for each $* \in S(R)$, if and only of R is a conducive domain and each overring of R is divisorial.*

The proof uses the following lemma.

Lemma 3.7. *Let R be an integral domain. If each semistar-operation is of finite character, then R is a TV-domain which is conducive and for each overring T of R, each semistar-operation on T is of finite character.*

Proof. Since $\bar{v} = (\bar{v})_f = \bar{t}$, then R is a TV-domain. Now, suppose that there is an overring T of R such that $T \subset K$ and $(R : T) = (0)$. Consider the semistar-operation $*$ on R defined by $A^* = A$ if $A \in F(R)$ and $A^* = K$, if $A \in \bar{F}(R)\backslash F(R)$. Clearly $T^* = K$. On the other hand, since $*$ is of finite character, then $T^* = \cup\{J^*/J \in f(R)$, and $J \subseteq T\}$. Since for each $J \in f(R) \subseteq F(R)$, $J^* = J$ then $T^* = \cup\{J/J \in f(R), J \subseteq T\} = T$, which is absurd. Hence R is a conducive domain.

Now, let T be an overring of R and let $* \in S(T)$. Consider the semistar-operation $*^\delta$ on R defined by $A^{*^\delta} = (AT)^*$ for each $A \in \bar{F}(R)$. Since $*^\delta = (*^\delta)_f$, then for each $A \in \bar{F}(T) \subseteq \bar{F}(R)$, $A^* = (AT)^* = A^{*^\delta} = A^{(*^\delta)_f} = \cup\{J^{*^\delta}/J \in f(R), J \subseteq A\} = \cup\{(JT)^*/J \in f(R), J \subseteq A\} \subseteq \cup\{I^*/I \in f(T), I \subseteq A\} = A^{*_f}$. Hence $* = *_f$, as desired. $\square$

Proof the Theorem. ($\Longrightarrow$) By Corollary 2.5, each semistar-operation on R is of finite character. By Lemma 3.7, R is a conducive domain. Now, let T be an overring of R, $\bar{v}_T$ the $\bar{v}$-semistar-operation on T and $*$ be the semistar-operation on R defined by $A^* = (AT)^{\bar{v}_T}$ for each $A \in \bar{F}(R)$. By hypothesis, $* = *_{\{R^*\}} = *_{\{T\}}$. So for each $A \in F(T) \subseteq F(R)$, $A^{v_T} = A^{\bar{v}_T} = A^* = A^{*_{\{T\}}} = AT = A$. Therefore T is divisorial.

($\Longleftarrow$) Let $*$ be a semistar-operation on R and set $T = R^*$. Then for each $A \in F(R)$, $A^{*_{\{T\}}} = AT = AR^* \subseteq (AR^*)^* = (AR)^* = A^*$. Now, since $T^* = T$ and T is divisorial, then the restriction $*|_{F(T)}$ of $*$ to $F(T)$ is trivial. Hence $(AT)^* = AT$. So $A^* \subseteq (AT)^* = AT = A^{*_{\{T\}}}$. Hence $A^* = A^{*_{\{T\}}}$ and therefore $* = *_{\{T\}}$. $\square$

Corollary 3.8. *Let R be a Prüfer domain. Then $S(R)=SFc(R)$ if and only if R is conducive and each overring of R is divisorial.*

Proof. ($\Longrightarrow$) Follows from Corollary 3.4 and Theorem 3.6.

($\Longleftarrow$) Let $* \in S(R)$ and set $T = R^*$. Since $T^* = R^{**} = R^* = T$, then $*|_{F(T)}$ is a star-operation on T. Since T is dividorial, then $*|_{F(T)}=d$. So for each $A \in F(T)$, $A^*=A$. Now, let $A \in F(R)$. Then $AT \in F(T)$. So $(AT)^*=AT$. Hence $A^*=(AR^*)^*=(AT)^*=AT=A^{*\{T\}}$. Since R is conducive and $K^* = K = K^{*\{T\}}$, then $* = *_{\{T\}}$. It follows that $S(R) = SFc(R)$. □

It is well-known that a valuation domain is divisorial if and only if its maximal ideal is principal and a maximal ideal of a valuation domain is either principal or idempotent. So each overring of a strongly discrete valuation domain (i.e. valuation domain with no idempotent prime ideals) is divisorial. Since valuation domains are conducive domains, the class of conducive domains with divisorial overrings contains the class of strongly discrete valuation domains. However, we present here an example of a non-integrally closed Noetherian conducive domain with divisorial overrings.

Example 3.9. Let k be a field and X an indeterminate over k. Set $R=k[[X^2, X^3]]$. Clearly R is a Noetherian local domain with maximal ideal $M=(X^2, X^3)$, $\bar{R}=R'=k[[X]]$, $M=X^2R'$ and $M^{-1}=(M : M)=k[[X]]=R'$. Since $(R : R')=M$ and R' is a *DVR*, by [4, Theorem 3.2], R is a conducive domain. We claim that $[R, qf(R)] = \{R, R', qf(R) = k((X))\}$. Indeed, let B be a proper overring of R, i.e. $R \subset B \subset qf(R)$. Since R is conducive, then $(R : B) \neq (0)$. So $R' = \bar{R} = \bar{B}$. Then $B \subseteq k[[X]]$. Let $f \in B\backslash R$ and write $f=a_0+a_1X+a_2X^2+a_3X^3+\cdots=a_1X+g$, with $g \in R$. Since $f \notin R$, then $a_1 \neq 0$. So $a_1^{-1} \in k$. Then $X = a_1^{-1}(f - g) \in B$. Then $R' = k[[X]] \subseteq B \subseteq R'$ and therefore $B = R'$. Since R' is a *DVR*, then it is divisorial. It suffices to show that R is divisorial. Let W be an R-module such that $R \subset W \subseteq R'$. Let $f \in W\backslash R$ and write $f=a_0+a_1X+a_2X^2+a_3X^3+\cdots=a_1X+g$ with $g \in R$. Since $f \notin R$, then $a_1 \neq 0$. So $a_1^{-1} \in k$. Since W is an R-module and $R \subset W$, then $X = a_1^{-1}(f - g) \in W$. Then $R' = k[[X]] \subseteq W \subseteq R'$ and therefore $W = R'$. So there is no proper R-module between R and R'. Now, let I be a nonzero ideal of R. Since R' is a *DVR*, then $IR' = fR'$ for some $f \in I$. Set $W = \{g \in R'/gf \in I\}$. It is easy to see that W is an R-module, $R \subseteq W \subseteq R'$ and $I = fW$. By the first part, either $R = W$ or $R' = W$. Hence $I = fR$ or $I = fR'$. If $I = fR$, then I is divisorial (as a principal ideal of R). If $I = fR'$, then $I^{-1} = (R : I) = (R : fR') = f^{-1}(R : R') = f^{-1}M$. Hence $I_v = (R : I^{-1}) = (R : f^{-1}M) = f(R : M) = fR' = I$. It follows that R is divisorial and so is each overring of R.

Corollary 3.10. *Let R be a completely integrally closed domain. Then $S(R) = SFc(R)$ if and only if R is a DVR.*

Proof. ($\Longrightarrow$) By Lemma 3.7, R is a conducive *TV*-domain. So, by [9, Theorem 2.3], R is a Krull domain. Now, by [4, Corollary 2.5], R is a *DVR*.

($\Longleftarrow$). Follows from [18, Theorem 48]. □

Corollary 3.11. *Let R be a Noetherian domain. Then* $S(R) = SFc(R)$ *if and only if R is conducive* (*so R is one-dimensional and local domain*).

Corollary 3.10 shows that for a Noetherian domain of dimension greater than 2, the inclusion $SFc(R) \subseteq S(R)$ is strict and a star-operation on R can have more than one extension to a semistar-operation on R.

4. About the cardinality of $SFc(R)$

In [15], Matsuda shows that for a nonlocal domain R, $4 + dim\ R \leqslant |S(R)|$ and he characterizes domains for which the equality holds. Our next result go to this way by substituting $SFc(R)$ to $S(R)$. We start this section by recalling the following results which will be often used in our proofs.

Lemma 4.1 (*Mimouni and Samman [17, Theorem 7]*). *Let R be a domain. Then* $|SFc(R)| = 1 + dim\ R$ *if and only if R is a valuation domain*.

Lemma 4.2 (*Mimouni and Samman [17, Theorem 10]*). *Let V be a valuation domain. Then* $S(V) = SFc(V)$, *that is, each semistar-operation on V is of finite character, if and only if V is strongly discrete*.

According to Jaballah [10] and by considering the extension $R \subset S$, we recall that the number $g(\text{Spec}(R, S))$, which permits to compute the number of intermediate rings between R and S, is obtained in the following way: For each vertex Q of the poset $\text{Spec}(R, S)$, let $\Gamma(Q)$ be the set of vertices covering Q. $P \in \Gamma(Q)$ if and only if $Q < P$ and there is no vertex U from $\text{Spec}(R, S)$ such that $Q < U < P$. For each vertex Q, we associate a number $g(Q)$ defined by:

$g(Q) = 1$ if $\Gamma(Q) = \emptyset$.

$g(Q) = \prod_{P \in \Gamma(Q)} (1 + g(P))$, if $\Gamma(Q) \neq \emptyset$.

$g(Spec(R, S)) = \prod_{Q \in Min(\text{Spec}(R,S))} g(Q)$, where $Min(\text{Spec}(R, S))$ is the set of minimal vertices of $Spec(R, S)$.

In the case where R is a Prüfer domain with finite spectrum and $S = K$, we have $|[R, K]| = g(\text{Spec}(R))$ [10, Corollary 3].

We are now ready to give the first theorem of this section.

Theorem 4.3. *Let R be a nonlocal domain. Then* $3 + dim\ R \leqslant |SFc(R)|$. *Moreover, the equality holds if and only if R is a Prüfer domain with exactly two maximal ideals M and N, and* $Spec(R)$ *is reduced to a unique Y-graph, that is,* $Spec(R) = \{(0) \subset P_1 \subset \cdots \subset P_{n-1} \subset M, N\}$ *and* $P_{n-1} \subseteq N$.

Proof. Assume that R is nonlocal. Set $n = dim\ R$ and let $(0) \subset P_1 \subset \cdots \subset P_{n-1} \subset M$ be a chain of prime ideals of R and N a maximal ideal of R with $M \neq N$. Then, by Corollary 2.5, $e = *_{\{K\}}$, $\bar{d} = *_{\{R\}}$, $*_{\{R_M\}}$, $*_{\{R_N\}}$ and $*_i = *_{\{R_{P_i}\}}$, for each $i \in \{1, \ldots, n-1\}$ are in $SFc(R)$ (note that if $n = 1$, then $\{e, \bar{d}, *_{\{R_M\}}, *_{\{R_N\}}\} \subseteq SFc(R)$). Hence $3 + dim\ R = n + 3 \leqslant |SFc(R)|$. Now, assume that $|SFc(R)| = 3 + dim\ R$. Then $n + 3 = |\{R, R_{P_1}, R_{P_2}, \ldots, R_{P_{n-1}}, R_M, R_N, K\}| \leqslant |[R, K]| \leqslant |SFc(R)| = n + 3$. So $[R, K] =$

$\{R, R_{P_1}, R_{P_2}, \ldots, R_{P_{n-1}}, R_M, R_N, K\}$, and $SFc(R) = \{*_{\{T\}}/T \in [R, K]\}$ (also if $n = 1$, clearly $[R, K] = \{R, R_M, R_N, K\}$). Since each overring of R is a localization of R, so flat over R, by [6, Theorem 1.1.1], R is a Prüfer domain. Clearly, $\text{Spec}(R) = \{(0) \subset P_1 \subset \cdots \subset P_{n-1} \subset M, N\}$. We wish to show that $P_{n-1} \subset N$. If $n = 1$, then $\text{Spec}(R) = \{(0), M, N\}$ is an *Y-graph*, as desired. So, we may assume that $n \geqslant 2$. Now, suppose that $P_{n-1} \not\subseteq N$. Two cases are then possible:

Case 1: For each $i \in \{1, \ldots, n-1\}$, $P_i \not\subseteq N$. Then $\text{Spec}(R)$ is of the form as shown in Fig. 1.

Since R is a Prüfer domain, then $|SFc(R)| = |[R, K]| = g(\text{Spec}(R)) = g(o) = (1 + g(P_1))(1 + g(N))$. Since $g(P_1) = n$ and $g(N) = 1$, then $n + 3 = |SFc(R)| = |[R, K]| = g(\text{Spec}(R)) = g(o) = (1 + g(P_1))(1 + g(N)) = 2(1 + n)$. Hence $n = 1$, which is a contradiction.

Case 2: There is $i \in \{1, \ldots, n-2\}$, such that $P_i \subseteq N$. We may assume that i is the great one. Then $\text{Spec}(R)$ is of the form as shown in Fig. 2.

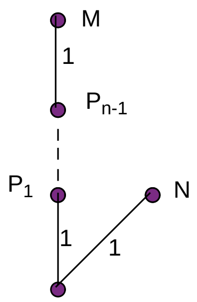

Fig. 1.

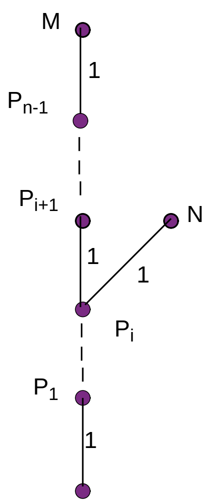

Fig. 2.

Hence $g(0) = (i + g(P_i))$. On the other hand $g(P_i) = (1 + g(N))(1 + g(P_{i+1}))$. Since $g(P_{i+1}) = n - i$, then $g(0) = i + 2(1 + n - i) = 2n + 2 - i$. Now, since $i \leqslant n - 2$, then $2 - n \leqslant -i$ and therefore $n + 4 = 2 - n + 2n + 2 \leqslant 2n + 2 - i = g(0) = 3 + n$, which is absurd. It follows that $P_{n-1} \subseteq N$ and therefore Spec(R) is reduced to the *Y-graph* $\{(0) \subset P_1 \subset \cdots \subset P_{n-1} \subset M \cap N\}$.

Conversely, since R is a Prüfer domain, then by Corollary 3.4, $|SFc(R)| = |[R, K]|$. By [10, Corollary 3], $|[R, K]| = g(0)$. Since Spec(R) is reduced to the unique *Y-graph* $\{(0) \subset P_1 \subset \cdots \subset P_{n-1} \subset M \cap N\}$, then $g(0) = n - 1 + g(P_{n-1})$. But $g(P_{n-1}) = (1 + g(M))(1 + g(N)) = 2.2 = 4$. Hence $g(0) = n + 3$, as desired.

By virtue of Theorem 4.3, if $|SFc(R)| \leqslant 2 + dim\ R$, then R is necessarily local. Also it is clear that $1 + dim\ R \leqslant |SFc(R)|$ and the equality holds if and only if R is a valuation domain [17, Theorem 7]. The following Theorem characterizes domains R for which $|SFc(R)| = 2 + dim\ R$.

Theorem 4.4. *Let R be a domain. Then $|SFc(R)| = 2 + dim\ R$ if and only if R is an fgv domain which is local, R' is a valuation domain, the extension $R \subseteq R'$ is minimal, and each overring of R is comparable to R'.*

Proof. ($\Longrightarrow$). By Theorem 4.3, R is local. Let $(0) \subset P_1 \subset \cdots \subset P_n = M$ be a chain of prime ideals of R, where $n = htM = dim\ R$. Then $\{R, R_{P_1}, \ldots, R_{P_{n-1}}, K\} \subseteq [R, K]$. If $\{R, R_{P_1}, \ldots, R_{P_{n-1}}, K\} = [R, K]$, then each overring of R is R-flat. So R is Prüfer [6, Theorem 1.1.1] and therefore R is a valuation domain, which is absurd by Lemma 4.1. Hence $\{R, R_{P_1}, \ldots, R_{P_{n-1}}, K\} \subset [R, k]$. Then $1 + n < |[R, k]| \leqslant |SFc(R)| = n + 2$ and therefore $|[R, K]| = n + 2$. Set $[R, K] = \{R, R_{P_1}, \ldots, R_{P_{n-1}}, K, T\}$. Hence $SFc(R) = \{\bar{d}, e, *_{\{R_{P_i}\}}, *_{\{T\}}\}$, with $i \in \{1, \ldots, n-1\}$. Since $\bar{t} \in SFc(R)$ and $R^{\bar{t}} = R$, the $\bar{d} = \bar{t}$. So $t = d$ and therefore R is an *fgv* domain. Now, if $R = R'$ then R is an *fgv* domain which is integrally closed, so Prüfer and hence a valuation domain, which is absurd. So $R \subset R'$. Since R is not a field, then R' cannot be a localization of R. Hence $R' = T$. Now, let $(0) \subset Q_1 \subset \cdots \subset Q_n = N$ be a chain of prime ideals of R', Such that $Q_i \cap R = P_i$. For each $i \in \{1, \ldots, n-1\}$, R'_{Q_i} is an overring of R. So $R'_{Q_i} = R_{P_j}$ for some $j \in \{1, \ldots, n-1\}$. Then $P_i = Q_i R'_{Q_i} \cap R = P_j R_{P_j} \cap R = P_j$. Hence $i = j$. It follows that $[R', K] = \{R', K, R'_{Q_i} / i \in \{1, \ldots, n-1\}\}$. Hence R' is a valuation domain and clearly the extension $R \subset R'$ is minimal and each overring of R is comparable to R', in fact $[R, K]$ is the chain $R \subset R' \subset R'_{Q_{n-1}} = R_{P_{n-1}} \subset \cdots \subset R'_{Q_1} = R_{P_1} \subset K$.

($\Longleftarrow$). Let $* \in SFc(R)$. If $R^* = K$, then $* = e$. If $R^* = R$, then the restriction $*|_{F(R)}$ is a star-operation of finite character on R. Since R is an *fgv* domain, then $*|_{F(R)} = d$. Now, let $A \in \bar{F}(R) \backslash F(R)$. Since $*$ is of finite character and for each $J \in f(R)$ with $J \subseteq A$, $J^* = J$, then $A^* = A$ and therefore $* = \bar{d}$. Then we may assume that $R \subset R^* \subset K$. By hypothesis, R' and R^* are comparable. Since the extension $R \subseteq R'$ is minimal and $R \subset R^*$, then $R' \subseteq R^*$. Since R' is a valuation domain, then $R^* = R'_Q$ for some nonzero prime ideal Q of R'. Now, we claim that $* = *_{\{R'_Q\}}$. Let $A \in \bar{F}(R)$ and $J \in f(R)$ such that $J \subseteq A$. Since R^* is a valuation domain, then $JR^* = aR^*$ for some nonzero a. Since $J \subseteq JR^*$, then $J^* \subseteq (JR^*)^* = (aR^*)^* = aR^{**} = aR^* = JR^* \subseteq AR^* = A^{*_{\{R^*\}}}$. Hence $A^* \subseteq A^{*_{\{R^*\}}}$.

Conversely, $A^{*_{\{R^*\}}} = AR^* \subseteq (AR^*)^* = (AR)^* = A^*$. Hence $A^{*_{\{R^*\}}} = A^*$ and therefore $* = *_{\{R^*\}} = *_{\{R'_Q\}}$. It follows that $SFc(R) = \{\bar{d}, e, *_{\{R'_Q\}}/Q$ nonzero prime ideal of $R'\}$. Since R' is a valuation domain and $dim\ \ R = dim\ \ R'$ is finite, then $|SFc(R)| = 2 + dim\ \ R$. □

Corollary 4.5. *Let R be a PVD. Then* $|SFc(R)|=2+dim\ \ R$ *if and only if* $[V/M : R/M]=2$, *where V is the valuation domain associated to R and M its maximal ideal*.

Proof. Let V be the valuation domain associated to R, M its maximal ideal and set $V/M = K$, $R/M = k$ and denotes by L the quotient field of R. In view of [1, Proposition 2.6], R is the pullback of the following diagram:

$$\begin{array}{ccc} R := \phi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \longrightarrow & V/M \end{array}$$

Assume that $|SFc(R)| = 2 + dim\ \ R$ and set $\text{Spec}(R) = Spec(V) = \{(0) \subset P_1 \subset \cdots \subset P_n = M\}$, where $n = dim\ \ R$. Then $SFc(R) = \{\bar{d}, e, *_{\{V\}}, *_{\{R_{P_i}\}}/i \in \{1, \ldots, n-1\}\}$. Since R is not a field, then R' cannot be a localization of R. Hence $R' = V$. Now the minimality of the extension $R \subseteq R' = V$ forces that $[K : k] = 2$. Conversely, if $[K : k] = 2$, then R is divisorial, $R' = V$, the extension $R \subseteq R' = V$ is minimal and each overring of R is comparable to V [2, Theorem 2.1 and Corollary 4.4]. By Theorem 4.4, $|SFc(R)| = 2 + dim\ \ R$. □

A local domain R with $|SFc(R)| = 2 + dim\ \ R$ is not, in general, a *PVD* as it shown by the following example.

Example 4.6 (*Mimouni and Samman [17, Example 18]*). Let k be a field, X an indeterminate over k and let $R = k[[X^2, X^3]]$. Then $R' = k[[X]]$ is a *DVR*, the extension $R \subseteq R'$ is minimal and clearly the only overrings of R are R, R' and $qf(R) = k((X))$. By Theorem 4.4, $|SFc(R)| = 3 = 2 + dim\ \ R$ and R is not a *PVD*.

Acknowledgements

I would like to express my sincere thanks to the referee for his/her helpful suggestions.

References

[1] D.F. Anderson, D.E. Dobbs, Pairs of rings with the same prime ideals, Canad. J. Math. 32 (1980) 362–384.
[2] E. Bastida, R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$, Michigan Math. J. 20 (1973) 79–95.
[3] S. Chapman, S. Glaz, Non-Noetherian Commutative Ring Theory, vol. 520, Kluwer Academic Publisher, Dordrecht, 2000.
[4] D. Dobbs, F. Fedder, Conducive integral domains, J. Algebra 86 (1984) 494–510.

[5] M. Fontana, J. Huckaba, in: S. Chapman, S. Glaz (Eds.), Localizing Systems and Semistar Operations, Non-Commutative Ring Theory, vol. 520, Kluwer Academic Publisher, Dordrecht, 2000, pp. 169–197.
[6] M. Fontana, J. Huckaba, I. Papick, Prüfer Domains, Marcel Dekker, New York, 1997.
[7] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, Inc., New York, 1972.
[8] W. Heinzer, Integral domains in which each non-zero ideal is divisorial, Mathematika 15 (1990) 164–170.
[9] E. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial, Michigan Math. J. 35 (1988) 291–300.
[10] A. Jaballah, On the number of intermediate rings, preprint.
[11] I. Kaplansky, Commutative Rings, Revised ed., University of Chicago Press, Chicago, 1974.
[12] R. Matsuda, Note on the number semistar-operations, Math. J. Ibaraki Univ. 31 (1999) 47–53.
[13] R. Matsuda, Note on valuation rings and semistar-operations, Comm. Algebra 25 (5) (2000) 2515–2519.
[14] R. Matsuda, Note on the number semistar-operations II, Far East J. Math. Sci. 2 (2000) 159–172.
[15] R. Matsuda, Note on the number semistar-operations III, Commutative Rings, Ayman Badawi, 2002, pp. 77–81.
[16] R. Matsuda, T. Sugatani, Semistar-operations on integral domains II, Math. J. Toyama Univ. 18 (1995) 155–161.
[17] A. Mimouni, M. Samman, Semistar-operations on valuation domains, Internat. J. Commutative Rings 2 (3) (2003) 131–141.
[18] A. Okabe, R. Matsuda, Semistar-operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21.
[19] M. Zafrullah, The v-operation and intersection of quotient of integral domains, Comm. Algebra 13 (1985) 1699–1712.