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Class semigroups of integral domains

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Abstract

This paper seeks ring-theoretic conditions of an integral domain R that reflect in the Clifford property or Boolean property of its class semigroup $\mathcal{S}(R)$, that is, the semigroup of the isomorphy classes of the nonzero (integral) ideals of R with the operation induced by multiplication. Precisely, in Section 3, we characterize integrally closed domains with Boolean class semigroup; in this case, $\mathcal{S}(R)$ identifies with the Boolean semigroup formed of all fractional overrings of R. In Section 4, we investigate Noetherian-like settings where the Clifford and Boolean properties of $\mathcal{S}(R)$ coincide with (Lipman and Sally–Vasconcelos) stability conditions; a main feature is that the Clifford property forces t-locally Noetherian domains to be one-dimensional Noetherian domains. Section 5 studies the transfer of the Clifford and Boolean properties to various pullback constructions. Our results lead to new families of integral domains with Clifford or Boolean class semigroup, moving therefore beyond the contexts of integrally closed domains or Noetherian domains.

1. Introduction

Let R be an integral domain. Following [43], we define the class semigroup of R, denoted $\mathcal{S}(R)$, to be the (multiplicative Abelian) semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals. The class semigroup of R contains, as subgroups, the class group $\mathrm{Cl}(R)$ and, hence, the Picard group $\mathrm{Pic}(R)$ of R.

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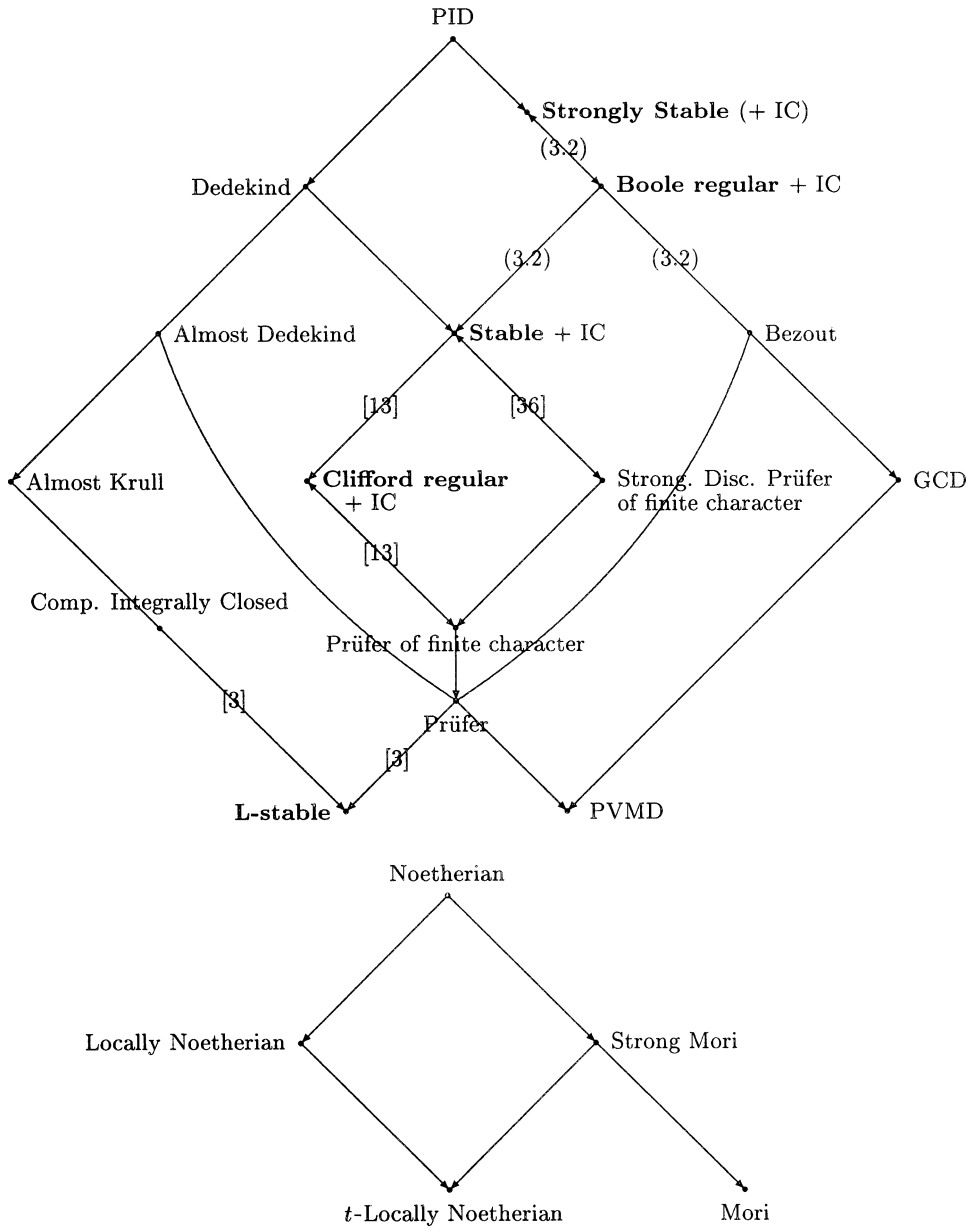
In 1994, Zanardo and Zannier [43] proved that if R is an integrally closed domain and $\mathcal{S}(R)$ is a Clifford semigroup then R is a Prüfer domain. The converse is not true since they showed that the ring of all entire functions in the complex plane (which is Bezout) fails to have this property. Their main result states that all orders in quadratic fields have Clifford class semigroup. In 1996, Bazzoni and Salce [14] investigated the structure of the class semigroup for a valuation domain V, stating that $\mathcal{S}(V)$ is a Clifford semigroup. In [10] and [11], Bazzoni examined the case of Prüfer domains of finite character, showing that these have Clifford class semigroup, too. Recently, she proved the converse in the case of integrally closed domains [13].

This paper aims at investigating ring-theoretic properties of an integral domain R which reflect in the Clifford property or the Boolean property of $\mathcal{S}(R)$. Precisely, in Section 3, our main theorem asserts that "*an integrally closed domain R has Boolean class semigroup if and only if R is a strongly discrete Bezout domain of finite character if and only if each nonzero ideal of R is principal in its endomorphism ring*." One may view this result as a satisfactory analogue of both [13, Theorem 4.5] on the Clifford property and [36, Theorem 4.6] on stability. As a prelude to this, we characterize valuation domains with Boolean class semigroup, stating that these are exactly the strongly discrete valuation domains [24]. Section 4 studies Noetherian-like contexts. We prove that "*if R is a t-locally Noetherian domain, then R has Clifford* (*resp., Boolean*) *class semigroup if and only if R is stable* (*resp., each nonzero ideal of R is principal in its endomorphism ring*)." In particular, t-locally Noetherian domains (such as Noetherian or strong Mori domains) with Clifford class semigroup turn out to be one-dimensional Noetherian domains. We also provide a characterization of Mori domains with Clifford or Boolean class semigroup that links them to stability, specifically, "*a Mori domain R is stable* (*resp., each nonzero ideal of R is principal in its endomorphism ring*) *if and only if R is a one-dimensional Clifford* (*resp., Boole*) *regular domain and the complete integral closure of R is Mori*." Section 5 treats the possible transfer of the Clifford and Boolean properties to pullbacks. New families of domains with Clifford or Boolean class semigroup stem from our results. Throughout, examples are provided to illustrate the scopes and limits of the results.

For the convenience of the reader, we summarize in the following two diagrams (see p. 622) the relations between the main classes of domains involved in this paper (where "+ IC" means that the implication requires the integrally closed hypothesis).

2. Preliminaries

Let us first recall the following definitions. A commutative semigroup S is said to be a *Clifford semigroup* if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x^2a = x$; and S is said to be *Boolean* if for each $x \in S$, $x = x^2$ (cf. [29]). The importance of a Clifford semigroup S resides in its ability to stand as a disjoint union of subgroups G_e, where e ranges over the set of idempotent elements of S, and G_e is the largest subgroup of S with identity equal to e. Often, the G_e's are called the constituent groups of S. Clearly, a semigroup S is Boolean if and only if the constituent groups of S are all trivial.



As in [13], we say that a domain R is *Clifford regular* if the class semigroup $\mathcal{S}(R)$ of R is a Clifford semigroup. By analogy with this, we say that a domain R is *Boole regular* if the class semigroup $\mathcal{S}(R)$ of R is a Boolean semigroup. At this point, recall Bazzoni's recent result [13, Theorem 4.5]: *an integrally closed domain R is Clifford regular if and only if R is a Prüfer domain of finite character* (i.e., each nonzero ideal is contained only in finitely many maximal ideals).

An ideal of an integral domain R is said to be L-*stable* (here L stands for Lipman) if $R^I = \bigcup (I^n : I^n) = (I : I)$, and R is called an L-*stable domain* if every nonzero ideal of R is L-stable [3]. Lipman [32] introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings (to give a characterization of Arf rings). In Lipman's context, *an integral domain R is* L-*stable if and only if R is Boole regular* (cf. [32, Lemma 1.11]).

An ideal I of an integral domain R is said to be *stable* if I is invertible in $(I : I)$, and R is called a *stable domain* provided each nonzero ideal of R is stable [3]. Sally and Vasconcelos [42] used this concept to settle Bass' conjecture on one-dimensional Noetherian rings with finite integral closure. Recall that a stable domain is L-stable [3, Lemma 2.1]. For recent developments on stability (in settings different than originally considered), we refer the reader to [3,13,36–38]. Of particular relevance to our study is Olberding's result [36, Theorem 4.6] stating that *an integrally closed domain R is stable if and only if R is a strongly discrete Prüfer domain of finite character*.

Throughout, all rings considered are integral domains. We shall use $\bar{I}$ to denote the isomorphy class of an ideal I.

We often will be appealing to the next results without explicit mention.

Lemma 2.1.

(1) *Let I be an ideal of an integral domain R. $\bar{I}$ is a regular element of $\mathcal{S}(R)$ if and only if $I = I^2(I : I^2)$* [10, Lemma 1.1].
(2) *A stable domain is Clifford regular* [13, Proposition 2.2].
(3) *A stable domain has finite character* [38, Theorem 3.3].
(4) *An integrally closed stable domain is Prüfer* [19, Lemma F].

The next lemma establishes the transfer of the Clifford and Boolean properties to two types of overrings.

Lemma 2.2. *Let R be an integral domain and B an averring of R. Assume that one of the following two assumptions holds*:

(a) *B is a flat extension of R,*
(b) *The conductor $(R : B)$ is nonzero.*

If R is a Clifford (*resp., Boole*) *regular domain, then so is B.*

Proof. (a) Let J be an ideal of B. It suffices to show that $J \subseteq J^2(J : J^2)$. Let $I := J \cap R$. By [39, Proposition 1.2(ii)], $J = IB$. For each $x \in (I : I^2)$, $xI^2 \subseteq I$ implies that $xI^2B \subseteq IB$. Hence $xJ^2 = x(IB)^2 = xI^2B \subseteq IB = J$. So $x \in (J : J^2)$ and hence $(I : I^2) \subseteq (J : J^2)$. Therefore $I = I^2(I : I^2) \subseteq J^2(J : J^2)$. So that $J \subseteq J^2(J : J^2)$.

(b) Assume that $(R : B) \neq 0$. Let $c \in (R : B)\backslash 0$, J an ideal of B, and $I = cJ$. Clearly, I is an ideal of R with $I^2(I : I^2) = cJ^2(J : J^2)$. Hence $cJ = I = I^2(I : I^2) = cJ^2(J : J^2)$. It follows that $J = J^2(J : J^2)$ and hence $\bar{J}$ is regular in $\mathcal{S}(B)$. Consequently, B is Clifford

regular. Now assume R is Boole regular. Here it suffices to notice that if $I^2 = qI$, then $J^2 = qJ$. □

Our next result, Proposition 2.3, will play a central role in the development of Sections 3 and 4. It generalizes Zanardo–Zannier's theorem mentioned in the introduction.

Proposition 2.3. *Let R be an integral domain. If R is a Clifford* (*resp., Boole*) *regular domain, then $\overline{R}$ is a Prüfer* (*resp., Bezout*) *domain, where $\overline{R}$ denotes the integral closure of R.*

Proof. The Clifford statement is handled by [41, Proposition 2.1] and [13, Proposition 2.3]. Next assume that R is a Boole regular domain. By the first part, $\overline{R}$ is a Prüfer domain. Let J be a finitely generated ideal of $\overline{R}$. Write $J = \sum_{i=1}^{i=r} a_i \overline{R}$. Let $T := R[a_1, \ldots, a_r]$ and $I := \sum_{i=1}^{i=r} a_i T$. Since T is a finitely generated R-module, then $(R : T) \neq 0$. By Lemma 2.2, $\mathcal{S}(T)$ is Boolean. So there is $0 \neq c \in K$ such that $I^2 = cI$. Since $I\overline{R} = J$, then $J^2 = cJ$. Hence $(J : J^2) = (J : cJ) = c^{-1}(J : J)$. Since J is invertible in $\overline{R}$, then $(J : J) = \overline{R}$, hence

$$c^{-1}\overline{R} = c^{-1}(J : J) = \big(J : J^2\big) = \big((J : J) : J\big) = \big(\overline{R} : J\big),$$

whence $c^{-1}J = J(\overline{R} : J) = \overline{R}$. So $J = c\overline{R}$ and thus $\overline{R}$ is a Bezout domain. □

Our first corollary characterizes almost Krull domains with Clifford or Boolean class semigroup. Notice that our elementary proof of this result does not appeal to [13, Theorem 4.5], rather it draws on basic properties of almost Krull domains.

Corollary 2.4. *Let R be an integral domain. Then R is almost Krull and Clifford* (*resp., Boole*) *regular if and only if R is Dedekind* (*resp., a PID*).

Proof. We just need to prove the "only if" assertion. Clearly, for any maximal ideal M of R, R_M inherits the Clifford property from R. Hence, by Proposition 2.3 R is an almost Dedekind domain. Suppose that there exists a nonzero ideal I of R which is not invertible, i.e., $II^{-1} \subsetneqq R$. Let $J := II^{-1}$. Then J is a proper trace ideal of R, hence $J^{-1} = (J : J) = R$ (since R is completely integrally closed), whence

$$\big(J : J^2\big) = \big((J : J) : J\big) = (R : J) = J^{-1} = R.$$

So

$$J = J^2\big(J : J^2\big) = J^2$$

(since J is regular in $\mathcal{S}(R)$). It follows that $J = J^n$ for each $n \geqslant 1$. Since R is almost Dedekind, $J = \bigcap_{n\geqslant 1}(J^n) = (0)$, the desired contradiction.

The Boolean statement follows from the Clifford statement and Proposition 2.3, completing the proof. □

A brief discussion at the end of Section 3 envisages a possible widening of the scope of Corollary 2.4 to completely integrally closed domains.

Corollary 2.5. *Let R be an integral domain and X an indeterminate over R. The following statements are equivalent*:

(i) *R is a field*;
(ii) *$R[X]$ is Boole regular*;
(iii) *$R[X]$ is Clifford regular*.

3. Boole regular domains

Clearly, a PID is Boole regular (see definition in Section 2) and a Boole regular domain is Clifford regular. Our purpose in this section is to characterize Boole regularity for integrally closed domains. Recall that the study of Clifford regularity—in the integrally closed context—was initiated in [10,11] and recently achieved in [13].

As a prelude, we characterize valuation domains with Boolean class semigroup, stating that these are exactly the strongly discrete valuation domains [24]. An integral domain is *strongly discrete* if it has no nonzero idempotent prime ideals. A stable domain trivially is strongly discrete.

We shall first find a natural stability condition that best suits the Boolean context. It can be termed as follows:

Definition 3.1. An integral domain R is called a *strongly stable domain* if each nonzero ideal of R is principal in its endomorphism ring $(I : I)$.

Next, we announce the main result of this section. First note that for any integral domain R, the set $\mathcal{F}_{\mathrm{OV}}(R)$ of fractional *overrings* of R is a Boolean semigroup with identity equal to R.

Theorem 3.2. *Let R be an integrally closed domain. The following statements are equivalent*:

(i) *R is a Boole regular domain*;
(ii) *R is a strongly discrete Bezout domain of finite character*;
(iii) *R is a strongly stable domain.*

Moreover, when any one condition holds, $\mathcal{S}(R) = \mathcal{F}_{\mathrm{OV}}(R)$, where $\overline{T}$ is identified with T for each fractional averring T of R.

The proof involves some preliminary results of independent interest.

Lemma 3.3. *Let R be an integral domain. The following statements are equivalent*:

(i) *R is a stable Boole regular domain*;
(ii) *R is a strongly stable domain.*

Proof. (i) $\Rightarrow$ (ii) Let I be a nonzero ideal of R. Since $\mathcal{S}(R)$ is Boolean, then $I^2 = cI$ for some $0 \neq c \in K$. So $(I : I^2) = (I : cI) = c^{-1}(I : I)$. Since R is stable, then $I(I : I^2) = (I : I)$. Hence $c^{-1}I = I(I : I^2) = (I : I)$ and therefore $I = c(I : I)$.

(ii) $\Rightarrow$ (i) Clearly, R is stable. Further, let I be a nonzero ideal of R. If $I = c(I : I)$, then $I^2 = cI$, as desired. □

Lemma 3.4. *Let R be an integrally closed domain. The following statements are equivalent*:

(i) *R is a strongly discrete Clifford regular domain*;
(ii) *R is a stable domain.*

Proof. By [36] we need only prove (i) $\Rightarrow$ (ii). This follows from a combination of [13, Theorem 4.5] and [36, Theorem 4.6]; however, we offer the following different elementary proof (which draws on the basic fact that the maximal ideal of a strongly discrete valuation domain is principal [24, Lemma 2.1]). Assume that (i) holds. By Proposition 2.3, R is a strongly discrete Prüfer domain. Let I be a nonzero ideal of R, $T := (I : I)$, and $J := I(T : I)$. Since $\bar{I}$ is regular in $\mathcal{S}(R)$, then $I = IJ$ and $J^2 = J$ [10, Proposition 2.1(1)]. Suppose that $J \subsetneqq T$. Let Q be a minimal prime ideal of T over J and $q = Q \cap R$. Then $T_Q = R_q$ is a strongly discrete valuation domain and hence $QT_Q = aT_Q$ for some $0 \neq a \in Q$. Since Q is minimal over J, then JT_Q is QT_Q-primary. So $JT_Q = (QT_Q)^r$ for some integer r. Since $J = J^2$, then $a^r T_Q = a^{2r} T_Q$, the desired contradiction. Therefore $J = T$ and hence R is stable. □

Recall that Bazzoni and Sake [14] proved that valuation domains have always Clifford class semigroup; next we characterize those among them with Boolean class semigroup.

Lemma 3.5. *Let V be a valuation domain. The following assertions are equivalent*:

(i) *V is a Boole regular domain*;
(ii) *V_P is a divisorial domain, for each nonzero prime ideal P of R*;
(iii) *V is a stable domain*;
(iv) *V is a strongly discrete valuation domain.*

Proof. (i) $\Rightarrow$ (ii) Claim: *If $\mathcal{S}(V)$ is Boolean, then V is a divisorial domain*. Indeed, let I be a nonzero ideal of V and $Z(V, I)$ the set of zero divisors of R modulo I. Then $Z(V, I) := P$ is a prime ideal of V and $(I : I) = V_P$. Since $\mathcal{S}(V)$ is Boolean, then there is $0 \neq c \in K$ such that $I^2 = cI$. Two cases are possible.

Case 1. $I(V_P : I) = V_P$. Then $I = aV_P$ for some nonzero $a \in I$. So

$$(V : I) = (V : aV_P) = a^{-1}(V : V_P) = a^{-1}P.$$

Hence

$$I_v = \big(V : (V : I)\big) = \big(V : a^{-1}P\big) = a(V : P).$$

Now, if P is not a maximal ideal of V, then $(V : P) = (P : P) = V_P$; hence $I_v = a(V : P) = aV_P = I$. So I is divisorial. If P is maximal in V, then $I = aV$. Here too, I is divisorial.

Case 2. $I(V_P : I) \subsetneqq V_P$. Since V_P is a *TP*-domain [22], then there is a prime ideal Q of V with $Q \subseteq P$ such that $I(V_P : I) = QV_P$. On the other hand, $I^2 = cI$ yields

$$(V_P : I) = \big(I : I^2\big) = (I : cI) = c^{-1}V_P.$$

So that

$$QV_P = I(V_P : I) = Ic^{-1}V_P = c^{-1}I,$$

whence $I = cQV_P$. So

$$V_P = (I : I) = (cQV_P : cQV_P) = (QV_P : QV_P) = V_Q.$$

It follows that $P = Q$ and $I = cQV_P = cPV_P = cP$. Since $I^2 = cI$, then $P = P^2$. Now P is a trace ideal of V. Then

$$(V : P) = (P : P) = V_P.$$

So

$$(V : I) = (V : cP) = c^{-1}(V : P) = c^{-1}V_P.$$

Therefore

$$I_v = \big(V : c^{-1}V_P\big) = c(V : V_P) = cP = I$$

and hence I is divisorial. Consequently, V is divisorial, completing the proof of our claim.

Now, let P be any nonzero prime ideal of V. By Lemma 2.2, V_P inherits the Boolean property from V. By the above claim, V_P is divisorial, as desired.

(ii) $\Rightarrow$ (iii) Let P be a prime ideal of V. By [27, Lemma 5.2], $P = PV_P = aV_P$ for some $a \in P$. By [3, Proposition 2.10], V is stable.

(iii) $\Rightarrow$ (i) Let I be a nonzero ideal of V and $P := Z(V, I)$. By (iii), I is invertible in $(I : I) = V_P$. Hence $I = aV_P$ for some $a \in I$. So $I^2 = aI$. Hence $\mathcal{S}(V)$ is Boolean.

(iii) $\Leftrightarrow$ (iv) is handled by [3, Proposition 2.10]. □

Notice that Lemma 3.5 gives rise to a large class of Boole regular domains that are not PIDs. Indeed, any strongly discrete valuation domain of dimension $\geqslant 2$ does (e.g., $k[X]_{(X)} + Yk(X)[[Y]]$, where k is a field and X, Y are indeterminates over k [24]).

Lemma 3.6. *An integrally closed domain R is locally Boole regular if and only if R is a strongly discrete Prüfer domain.*

Proof. Combine Proposition 2.3 and Lemma 3.5. □

Lemma 3.7. *An integrally closed domain R is Boole regular if and only if R is a stable Bezout domain.*

Proof. Assume R is Boole regular. By Proposition 2.3, R is Bezout. Further, a combination of Lemmas 2.2 and 3.6 ensures that R is a strongly discrete Prüfer domain. It turns out that R is a strongly discrete Clifford domain, hence it is stable by Lemma 3.4. Conversely, Let I be an ideal of R. Then $T := (I : I)$ is a Bezout domain. Further, I is invertible in T, so it is principal in T to complete the proof. □

Proof of Theorem 3.2. (i) $\Rightarrow$ (ii) Follows from Lemma 3.7 along with the facts that a stable domain is necessarily strongly discrete and has finite character.

(ii) $\Rightarrow$ (i) Follows from [36, Theorem 4.6] (and Lemma 3.7); however, we offer the following direct proof which draws on Bazzoni's study of the groups associated to idempotents in the class semigroup. Next, assume that R is a strongly discrete Bezout domain of finite character. Then $\mathcal{S}(R) = \bigvee G_{\bar{J}}$, where $\bar{J}$ ranges over the set of idempotent elements of $\mathcal{S}(R)$. By [11, Theorem 3.1], an element $\bar{J}$ of $\mathcal{S}(R)$ is idempotent if and only if there exists a unique nonzero idempotent fractional ideal L of R such that $J \cong L$ and L satisfies one of the following two conditions:

(1) $L = T$, where T is a fractional overring of R, or
(2) $L = P_1 P_2 \cdots P_n T$, where each P_i is a nonzero idempotent prime ideal of R and T is a fractional overring of R.

Since R is strongly discrete, then there is no nonzero idempotent prime ideals. This rules out the L's issued from the second condition. Further, by [12, Proposition 2.2], the group $G_{\overline{T}}$ associated to $\overline{T}$ coincides with the class group $\mathrm{Cl}(T)$ for each fractional overring T of R. Since R is Bezout, then each overring T of R is Bezout and therefore $\mathrm{Cl}(T)$ is trivial. Hence the constituent groups of $\mathcal{S}(R)$ are all trivial, whence $\mathcal{S}(R)$ is Boolean, as desired.

(i) $\Leftrightarrow$ (iii) is handled by Lemmas 3.3 and 3.7.

Finally, assume that (i)–(iii) hold. Clearly, $\mathcal{S}(R) = \{\overline{T}\colon\ T \in \mathcal{F}_{\mathrm{OV}}(R)\}$ by [11, Theorem 3.1] mentioned above. Moreover, due to the uniqueness required by this theorem, one can identify $\overline{T}$ with T for each $T \in \mathcal{F}_{\mathrm{OV}}(R)$, leading therefore to the identification of $\mathcal{S}(R)$ with the Boolean semigroup $\mathcal{F}_{\mathrm{OV}}(R)$, completing the proof of the theorem. □

Example 3.8. In [33, Construction 1], Loper shaped an example of a generalized Dedekind domain (hence a strongly discrete Prüfer domain [23]) which is not Bezout. Further, (one can easily check that) it has finite character. Hence it is stable [36] but not Boole regular (Theorem 3.2). It follows that Theorem 3.2 does not extend to strongly discrete Prüfer domains of finite character (equivalently, integrally closed stable domains).

Remark 3.9. Theorem 3.2 and its satellite lemmas yield immediate consequences:

(1) Unlike Clifford regularity, Boole regularity is not a local property for the class of integrally closed domains of finite character.
(2) If R is an integrally closed domain that is Boole regular (equivalently, strongly stable), then so is any overring of R.
(3) Stability and strong stability do not coincide in general (e.g., Dedekind domains that are not PIDs). They do however in integrally closed semilocal contexts (see Corollary 3.10).
(4) Unlike stability, strong stability is not a local property for the class of domains of finite character.
(5) If R is a strongly stable domain, then so is its integral closure $\overline{R}$.

Moreover, a Bezout domain of finite character need not be Boole regular (e.g., valuation domains with nonzero idempotent prime ideals). Consequently, in view of the above discussion, Theorem 3.2 may stand as a satisfactory analogue of both [13, Theorem 4.5] and [36, Theorem 4.6] for Boole regularity and strong stability, respectively.

In the semilocal context where "Prüfer" elevates to "Bezout", most of the notions in play collapse, as shown by the next result.

Corollary 3.10. *Let R be an integrally closed semilocal domain. The following statements are equivalent*:

(i) *R is a strongly stable domain*;
(ii) *R is a Boole regular domain*;
(iii) *R is a stable domain*;
(iv) *R is a strongly discrete Clifford regular domain*;
(v) *R is a strongly discrete Prüfer domain.*

It is worth noticing that from Corollary 3.10 stems a large family of examples of integrally closed Boole regular domains that are neither PIDs nor strongly discrete valuation domains (e.g., semilocal strongly discrete Prüfer domains of dimension $\geqslant 2$). Recall that the class of strongly discrete Prüfer domains of finite character properly contains the class of integrally closed Boole regular domains.

We close this section with a brief discussion of the completely integrally closed case. Indeed, by Theorem 3.2, *a completely integrally closed domain is Boole regular if and only if it is a PID.* This extends the Boolean statement of Corollary 2.4. However, a one-dimensional completely integrally closed Clifford regular domain (e.g., a non-discrete rank-one valuation domain) need not be Dedekind. Compare to the Clifford statement of Corollary 2.4 as well as to the known fact that a one-dimensional integrally closed stable domain is Dedekind.

4. Noetherian-like settings

This section investigates the class semigroup for two large classes of Noetherian-like domains, that is, t-locally Noetherian domains and Mori domains. Precisely, we study conditions under which stability and strong stability characterize Clifford regularity and Boole regularity, respectively. A main feature of our first theorem is that Clifford regularity forces the Noetherianity of t-locally Noetherian domains. However, the second main theorem (on Mori domains) may allow one, a priori, to move beyond the context of Noetherian domains. Unfortunately, we are not able to shape an example that supports this claim. (See the brief discussion at the end of this section.)

In order to provide some background for the present section, we review some terminology related to star-operations [26]. Let R be an integral domain. For a nonzero fractional ideal I of R, set $I_v := (I^{-1})^{-1}$; $I_t := \bigcup J_v$ where J ranges over the set of finitely generated fractional ideals of R contained in I; and $I_w := \bigcup (I : J)$ where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$. We say that I is *divisorial* if $I_v = I$; a *t-ideal* if $I_t = I$; and a *w-ideal* if $I_w = I$. Any divisorial ideal is a w-ideal. Now, R is said to be a *Mori domain* if it satisfies the ascending chain condition on divisorial ideals [5,6,8,25] and a *strong Mori domain* if it satisfies the ascending chain condition on w-ideals [20,35]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori.

Finally, we say that R is *t-locally Noetherian* if R_M is Noetherian for each maximal t-ideal M of R [30]. Recall that strong Mori domains are t-locally Noetherian [20, Theorem 1.9].

Throughout, we shall use $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$, and $t\text{-Max}(R)$ to denote the sets of prime ideals, maximal ideals, and maximal t-ideals, respectively, of R.

We begin by providing necessary t-ideal-theoretic conditions for Clifford regularity.

Lemma 4.1. *Let R be a Clifford regular domain. Then $I_t \subsetneqq R$ for each nonzero proper ideal I of R. In particular,* $\mathrm{Max}(R) = t\text{-Max}(R)$.

Proof. Deny. Then there exists a nonzero proper finitely generated ideal I of R such that $I_v = R$. So $(I : I) = I^{-1} = R$. Hence $(I : I^2) = ((I : I) : I) = (R : I) = I^{-1} = R$. Since $\bar{I}$ is regular in $\mathcal{S}(R)$, then $I = I^2(I : I^2) = I^2$, a contradiction by [31, Theorem 76]. □

Next, we state our first theorem of this section.

Theorem 4.2. *Let R be a t-locally Noetherian domain. Then R is Clifford* (*resp., Boole*) *regular if and only if R is stable* (*resp., strongly stable*). *Moreover, when any one condition holds, R is either a field or a one-dimensional Noetherian domain.*

Proof. Assume that R is Clifford regular. By Lemma 4.1, we have $\mathrm{Max}(R) = t\text{-Max}(R)$. Hence R is locally Noetherian. Now, suppose that R is not stable. Then there is a nonzero ideal I of R such that $I(T : I) \subsetneqq T$, where $T := (I : I)$. So there is a maximal ideal M of R containing I such that $(I(T : I))_M \subsetneqq T_M \subseteq (I_M : I_M)$. Set $J := I_M(I_M : I_M^2)$. By [13, Proposition 2.9], $J = (I(T : I))_M$. So J is a nonzero proper ideal of $(I_M : I_M)$. Since

$\mathcal{S}(R_M)$ is Clifford, then $\overline{I_M}$ is regular in $\mathcal{S}(R_M)$. So $I_M = I_M^2(I_M : I_M^2) = I_M J$. Since R_M is Noetherian, then I_M is a f.g. ideal of R_M and therefore a f.g. ideal of $(I_M : I_M)$. By [31, Theorem 76], $J = (I_M : I_M)$, the desired contradiction. The converse is handled by Lemma 2.1.

The Boolean statement follows from the Clifford statement and Lemma 3.3.

Finally, one may assume that R is a stable domain that is not a field. Then R has finite character and hence is locally Noetherian by Lemma 4.1. So R is Noetherian by [26, Lemma 37.3]. Further, we have $\dim(R) = 1$ by [3, Proposition 2.4], completing the proof of the theorem. □

Thus, a strong Mori domain that is Clifford regular (equivalently, stable) is necessarily a Noetherian domain. Here, Clifford regularity forces the w-operation to be trivial (see also [35, Proposition 1.3]). Also noteworthy is that while *a t-locally Noetherian stable domain is necessarily a one-dimensional* L*-stable domain*, the converse does not hold in general. For instance, consider an almost Dedekind domain which is not Dedekind and appeal to Corollary 2.4. However, the equivalence holds for Noetherian domains:

Corollary 4.3 ([13, Theorem 2.1] and [3, Proposition 2.4]). *Let R be a Noetherian domain that is not a field. The following statements are equivalent*:

(i) *R is Clifford regular*;
(ii) *R is stable*;
(iii) *R is* L*-stable with* $\dim(R) = 1$.

Corollary 4.4. *Let R be a local Noetherian domain such that the extension $R \subseteq \overline{R}$ is maximal, where $\overline{R}$ denotes the integral closure of R. The following statements are equivalent*:

(i) *R is Boole regular*;
(ii) *R is strongly stable*;
(iii) *R is stable and $\overline{R}$ is a PID.*

Proof. In view of Theorem 4.2 and Proposition 2.3, we need only prove the implication (iii) ⇒ (ii). Let I be a nonzero ideal of R and $T := (I : I)$. Since here $\overline{R}$ is identical to_the complete integral closure of R, then $R \subseteq T \subseteq \overline{R}$, hence either $R = T$ or $T = \overline{R}$. If $R = T$, then I is invertible and hence principal in R (since R is local). If $T = \overline{R}$, the conclusion is trivial. □

Corollary 4.4 generates new families of Boole regular domains (i.e., with regard to those integrally closed provided by Lemma 3.5 and Corollary 3.10).

Example 4.5. Let k be a field and X an indeterminate over k. Let $R := k[X^2, X^3]_{R\setminus(X^2,X^3)}$. Clearly, $\overline{R} = k[X]_{R\setminus(X^2,X^3)}$ is a PID and the extension $R \subseteq \overline{R}$ is maximal. Further, R is a Noetherian Warfield domain, hence stable (cf. [15]). Consequently, R is a one-dimensional non-integrally closed local Noetherian domain that is Boole regular by Corollary 4.4.

At this point, note that a Noetherian domain that is Clifford regular (equivalently, stable) need not be Boole regular (equivalently, strongly stable). For instance, consider Dedekind domains that are not PIDs (cf. Remark 3.9). The following is an example of a non-integrally closed Noetherian Clifford regular domain that is not Boole regular. It also shows that Corollary 4.4 fails, in general, when R is no longer local.

Example 4.6. Under the same notation of the above example, let $R := k[X^2, X^3]$. Clearly, $\overline{R} = k[X]$ and the extension $R \subseteq \overline{R}$ is maximal. Similarly, R is stable (and hence Clifford regular). However, R is not Boole regular since the ideal $I := (X^2 - 1, X^3 - 1)$ is not principal in $(I : I) = R$.

We now aim toward a possible characterization of Mori domains with Clifford or Boolean class semigroup that links them to stability. In what follows, we shall use $\overline{R}$ and R^* to denote the integral closure and complete integral closure, respectively, of an integral domain R. Suitable background on Mori domains is [6].

Next, we announce our second theorem of this section.

Theorem 4.7. *Let R be a Mori domain. The following statements are equivalent*:

(i) *R is a one-dimensional Clifford* (*resp., Boole*) *regular domain and R^* is Mori*;
(ii) *R is stable* (*resp., strongly stable*).

The proof requires the following result which provides a classification for Mori stable domains.

Lemma 4.8. *Let R be an integral domain. The following statements are equivalent*:

(i) *R is a Mori stable domain*;
(ii) *R has finite character and R_M is a DVR or a one-dimensional Mori stable domain for each $M \in \operatorname{Max}(R)$.*

Proof. Combine [37, Corollary 2.7] and [25, Theorem 4.18]. □

Proof of Theorem 4.7. (i) $\Rightarrow$ (ii) By Proposition 2.3, $\overline{R}$ is a Prüfer domain. It follows that R^* is a Dedekind domain. Further, $\dim(R) = 1$ implies that $\dim_v(R) = 1$ by [1, Theorem 1.10], where $\dim_v(R)$ denotes the valuative dimension of R. Now, let I be a nonzero proper ideal of R. Set $B := (I : I)$ and $J := I(B : I)$. Suppose that J is a *proper* ideal of B. Since $R \subseteq B \subseteq R^*$, then $1 = \dim_v(R) \geqslant \dim_v(B) \geqslant \dim(B) \geqslant 1$, whence $\dim(B) = 1$. Let P be a prime ideal of B such that $J \subseteq P$. So $\operatorname{ht} P = 1$. By [8, Proposition 1.1], there exists a prime ideal Q of $B^* = R^*$ such that $Q \cap B = P$. Since $\bar{I}$ is regular in $\mathcal{S}(R)$, then $I = I^2(I : I^2) = I^2((I : I) : I) = I^2(B : I) = IJ$. Hence $B = (I : I) = (I : IJ) = ((I : I) : J) = (B : J) = (J : J)$ (since J is a trace ideal of B). So $(J : J^2) = ((J : J) : J) = (B : J) = B$. Hence $J^2(J : J^2) = J^2$. Since $(R : B) \neq (0)$, B is Clifford regular by Lemma 2.2. So that $J = J^2(J : J^2) = J^2$, hence $JR^* = J^2R^* =$

$(JR^*)^2$. Since R^* is a Noetherian domain, $JR^* = R^*$ by [31, Theorem 76], whence $R^* = JR^* \subseteq PR^* \subseteq Q$, absurd. Therefore $J = B$ and hence I is stable.

The Boolean statement follows from the Clifford statement and Lemma 3.3 to complete the proof of the forward direction.

(ii) $\Rightarrow$ (i) Lemma 4.8 yields $\dim(R) = 1$. It remains to show that R^* is Mori, equivalently, Dedekind. Recall first that every overring of a stable domain is stable [38, Theorem 5.1]. Thus, $\overline{R}$ is now a one-dimensional integrally closed stable domain. Hence $\overline{R}$ is Dedekind and so is R^*, completing the proof of the theorem. □

It is worth recalling that for a Noetherian domain R we have: "$\dim(R) = 1$ if and only if $\dim(R^*) = 1$ if and only if R^* is Dedekind" (since here $R^* = \overline{R}$). The same result holds if R is a Mori domain such that $(R : R^*) \neq 0$ [8, Corollaries 3.4(1) and 3.5(1)]. Also it was stated that the "only if" assertion holds for seminormal Mori domains [8, Corollary 3.4(2)]. However, beyond these contexts, the problem remains elusively open. This explains the cohabitation of "$\dim(R) = 1$" and "R^* is Mori" hypotheses in Theorem 4.7. In this vein, we set the following open question: "*Let R be a local Mori Clifford regular domain. Is* $\dim(R) = 1$ *if and only if R^* is Dedekind?*"

Next, we announce our third theorem of this section. It partly draws on Theorem 4.7 and treats two well-studied large classes of Mori domains [6]. Recall that a domain R is seminormal if $x \in R$ whenever $x \in K$ and $x^2, x^3 \in R$ (equivalently, $x^n \in R$ for all $n \gg 0$).

Theorem 4.9. *Let R be a Mori domain. Assume that either* (a), (b), *or* (c) *holds*:

(a) *The conductor $(R : R^*) \neq 0$;*
(b) *R is seminormal*;
(c) *The extension $R \subseteq R^*$ has at most one proper intermediate ring.*

Then R is a Clifford (*resp., Boole*) *regular domain if and only if R is a stable* (*resp., strongly stable*) *domain.*

The proof of (c) requires the following technical lemma.

Lemma 4.10. *Let R be a Clifford regular domain and let I be a nonzero ideal of R. If $(I : I)$ is a Mori domain, then I is a stable ideal of R.*

Proof. Assume $T := (I : I)$ is a Mori domain. By Lemma 2.2, T is Clifford regular. Suppose that I is not stable. Then $J := I(T : I)$ is a proper trace ideal of T. Since $\bar{I}$ is regular in $\mathcal{S}(R)$, then $I = I^2(I : I^2) = IJ$. So $T = (I : I) = (I : IJ) = ((I : I) : J) = (T : J) = (J : J)$. Hence $J_v = T$. Since T is Mori, then $J_t = J_v = T$ (the v- and t-operations being with respect to T). Lemma 4.1 leads to the desired contradiction. □

Proof of Theorem 4.9. We need only prove the "only if" assertion for Clifford regularity. Let R be a Mori Clifford regular domain that is not a field. By Proposition 2.3, R^* is a Prüfer domain.

(a) Assume $(R : R^*) \neq 0$. By [4, Corollary 18], R^* is a Krull domain and thus Dedekind, so that $\dim(R^*) = 1$. It follows that $\dim(R) = 1$ by [8, Corollary 3.4]. Theorem 4.7 leads to the conclusion.

(b) Assume that R is seminormal. According to [5, Theorem 2.9], R^* is a Krull domain and hence Dedekind. In view of Theorem 4.7, we need only show that $\dim(R) = 1$. Let M be any maximal ideal of R. Clearly, R_M is a seminormal local Mori Clifford regular domain. Therefore, we may assume that R is local with maximal ideal M. Suppose that $\operatorname{ht} M = \dim(R) \geqslant 2$. By Lemma 4.1, M is a t-ideal of R. Since R is Mori, then $M_v = M_t = M$. Hence $R \subsetneqq M^{-1}$. By [40, Proposition 1], M is strongly divisorial, so that $T := (M : M) = M^{-1}$ is a Mori domain. Since R is seminormal, by [8, Lemma 2.5] there is a non-divisorial prime Q of T contracting on M such that $\operatorname{ht}_T Q \geqslant 2$. Since Q is not divisorial in T, $(T : Q) = T$ by [40, Proposition 1], whence $Q_t = Q_v = T$ (the t- and v-operation being with respect to T). Further, T is Clifford regular by Lemma 2.2. Therefore, Lemma 4.1 yields the desired contradiction. Hence $\dim(R) = 1$, as desired.

(c) Assume that $R \subseteq R^*$ has at most one proper intermediate ring. Let I be a nonzero ideal of R and let $J := II^{-1}$. Since $R \subseteq (I : I) \subseteq R^*$, then either $(I : I) = R^*$, $R = (I : I)$, or $R \subsetneqq (I : I) \subsetneqq R^*$. In view of (a) and Lemma 4.10, we need only handle the late case. Since now $R \subsetneqq (I : I) \subseteq (J : J) = J^{-1} \subseteq R^*$, then either $(J : J) = J^{-1} = R^*$ or $(I : I) = (J : J) = J^{-1}$. The former case follows from (a). The latter case follows from Lemma 4.10, since J^{-1} is a Mori domain by [34, Theorem 11]. Consequently, in all cases I is stable and so is R. □

One may wonder about the existence of (one-dimensional) Mori stable domains that are not Noetherian. Indeed, the pullback construction—a main source for non-Noetherian non-Krull Mori domains—can be of no help in this regard. More precisely, let T be a domain, M a maximal ideal of T, K its residue field, $\phi : T \to K$ the canonical surjection, and D a proper subring of K with quotient field $\operatorname{qf}(D) = k$. Let $R := \phi^{-1}(D)$. Then R is a Mori stable domain only if $R = T$. This follows easily from a combination of [25, Theorem 4.18] and [37, Theorem 2.6] (i.e., while the former result yields $D = k$, the latter, applied to (T_M, R_M, MR_M), yields $k = K$).

Also, it turns out that non-Noetherian Mori Clifford regular domains cannot stem from our results on pullbacks (Section 5). Indeed, under the hypotheses of Theorem 5.1(2) below, Noetherianity and the Mori property coincide for the pullback R.

5. Pullbacks

The purpose of this section is to characterize Clifford regularity and Boole regularity in pullback constructions. Our work is motivated by an attempt to generating new families of integral domains with Clifford or Boolean class semigroup, moving therefore beyond the classical contexts of integrally closed or Noetherian domains.

Let us fix the notation for the rest of this section. Let T be an integral domain, M a maximal ideal of T, K its residue field, $\phi : T \to K$ the canonical surjection, D a proper

subring of K, and $k := \mathrm{qf}(D)$. Let $R := \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

First, we wish to shed light on some features imposed by a possible passage of Clifford regularity to pullbacks. As a matter of fact, R need not be Clifford regular even when D is a PID with $k = K$ and T is a DVR (e.g., $R := \mathbb{Z} + X\mathbb{Q}[[X]]$) or when $D = k$ and T is local (see Example 5.3). In the well-studied case where T is integrally closed (e.g., a valuation domain or a polynomial ring over a field), Clifford regularity of R transfers to $\overline{R}$, since here $\overline{R} = \phi^{-1}(D')$, where D' is the integral closure of D in K. Further, $\overline{R}$ and (hence) R have finite character, which forces D to be semilocal. This follows easily from a combination of Lemma 2.2, Proposition 2.3, the finite character requirement [13, Theorem 4.5], and the well-known fact that $\mathrm{Spec}(R)$ is an amalgamated sum of $\mathrm{Spec}(D)$ and $\mathrm{Spec}(T)$ over the conductor M [21].

Next, we announce our first theorem of this section. It particularly provides a necessary and sufficient condition for a pseudo-valuation domain (i.e., PVD) to inherit Clifford or Boole regularity.

Theorem 5.1. *Under the above notation, the following hold*:

(1) *If R is Clifford* (*resp., Boole*) *regular, then so are T and D, and* $[K : k] \leqslant 2$.
(2) *Assume $D = k$ and T is a valuation* (*resp., strongly discrete valuation*) *domain.*

Then R is Clifford (*resp., Boole*) *regular if and only if* $[K : k] = 2$.

We need the following technical lemma.

Lemma 5.2. *Under the above notation, let W be a D-submodule of K containing D. Then* $\phi^{-1}(W : W) = (\phi^{-1}(W) : \phi^{-1}(W))$.

Proof. Let W be a D-module such that $D \subseteq W \subsetneq K$. Since $1 \in W$, then $(W : W) \subseteq W$. So $\phi^{-1}(W : W) \subseteq \phi^{-1}(W) \subseteq T$. Now, let $x \in \phi^{-1}(W : W)$. So, for each $z \in \phi^{-1}(W)$, $\phi(xz) = \phi(x)\phi(z) \in W$. Then $xz \in \phi^{-1}(W)$ and therefore $x \in (\phi^{-1}(W) : \phi^{-1}(W))$. Conversely, let $x \in (\phi^{-1}(W) : \phi^{-1}(W))$. Since $1 \in \phi^{-1}(W)$, then $x \in \phi^{-1}(W) \subseteq T$ and $x\phi^{-1}(W) \subseteq \phi^{-1}(W)$ implies that $\phi(x)W = \phi(x\phi^{-1}(W)) \subseteq \phi(\phi^{-1}(W)) = W$. Hence $\phi(x) \in (W : W)$, as desired. □

Proof of Theorem 5.1. (1) Assume that R is Clifford (resp., Boole) regular. Then so is T by Lemma 2.2. Let J be a nonzero (integral) ideal of D and let $I := \phi^{-1}(J)$. By [28, Proposition 6], $(I : I^2) = \phi^{-1}(J : J^2)$. So $J = \phi(I) = \phi(I^2(I : I^2)) = J^2(J : J^2)$ and therefore D is Clifford regular. Now, assume that R is Boole regular. Then there exists

$0 \neq c \in \mathrm{qf}(R)$ such that $I^2 = cI$. Since J is nonzero, then $M \subsetneqq I$. Let $R_0 = \phi^{-1}(k)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Since $M \subsetneqq I \subseteq IR_0$ and M is a maximal ideal of R_0, then $IR_0 = R_0$. So

$$1 = \sum_{i=1}^{i=n} a_i x_i,$$

where $a_i \in I$ and $x_i \in R_0$ for each i. Hence

$$c = \sum_{i=1}^{i=n} c a_i x_i.$$

Since $ca_i \in cI = I^2 \subseteq R \subseteq R_0$, then $ca_i x_i \in R_0$ for each i, hence $c \in R_0$. So $\phi(c) \in k = \mathrm{qf}(D)$ and $J^2 = \phi(c)J$. It follows that D is Boole regular. It remains to prove that $[K : k] \leqslant 2$. Notice first that R_0 is Clifford by Lemma 2.2.

Step 1. We claim that, for each $x \in K$, $x^2 \in k + xk$. By a contrast way, suppose there exists $x \in K$ such that $x^2 \notin k + xk$. Let W be the k-vector space denned by $W := k + xk$ and let I be the ideal of R_0 given by $I := m\phi^{-1}(W)$ for some nonzero $m \in M$. We first show that $(W : W) = k$. It is clear that $k \subseteq (W : W)$. Since $1 \in W$, then $(W : W) \subseteq W$. Let $z \in (W : W)$. Write $z = a + bx$, where $a, b \in k$. Since $x \in W$, then $zx \in W$. So $bx^2 + ax = zx = c + dx$ for some $c, d \in k$. If $b \neq 0$, then $x^2 = b^{-1}(d - a)x + b^{-1}c \in k + xk$, which is absurd. So $b = 0$ and therefore $z = a \in k$. Hence $(W : W) = k$. Now, by Lemma 5.2,

$$(I : I) = \big(m\phi^{-1}(W) : m\phi^{-1}(W)\big) = \big(\phi^{-1}(W) : \phi^{-1}(W)\big) = \phi^{-1}\big((W : W)\big) = \phi^{-1}(k) = R_0.$$

So

$$\big(I : I^2\big) = \big((I : I) : I\big) = (R_0 : I) = m^{-1}\phi^{-1}\big((k : W)\big) = m^{-1}\phi^{-1}(0) = m^{-1}M.$$

Hence $I^2(I : I^2) \subseteq mM \subsetneqq I$, which is a contradiction since $\bar{I}$ is regular in $\mathcal{S}(R_0)$. It follows that for each $x \in K \backslash k$, $[k(x) : k] = 2$.

Step 2. Suppose that $[K : k] \geqslant 3$. Consider a free system $\{1, x, z\}$ of K as a k-vector space. Let $W := k + xk + zk$ and $I := m\phi^{-1}(W)$ for some nonzero $m \in M$. We wish to show that $(W : W) = k$. Let $y \in (W : W) \subseteq W$. Write $y = a + bx + cz$. Since $x \in W$,

then $xy = ax + bx^2 + czx \in W$. By the first step, $x^2 = dx + e$ for some $d, e \in k$. Hence $ax + bdx + be + cxz = xy \in W$. So $cxz = xy - (a + bd)x - be \in W$. If $c \neq 0$, then $xz \in W$, whence W is a ring. So $W = k[x, z] = k(x, z)$ (since, by the first step, x and z are algebraic over k). Hence $[W : k] = [k(x, z) : k] = 4$ which is absurd. It follows that $c = 0$. Similarly, using the fact that $z \in W$, we obtain that $b = 0$. Hence $y = a \in k$ and therefore $(W : W) = k$. Now, as in the first step, we obtain that $I^2(I : I^2) \subseteq mM \subsetneq I$, which is a contradiction. It follows that $[K : k] = 2$.

(2) Assume that $D = k$ and $[K : k] = 2$. Let I be a nonzero (integral) ideal of R. If I is an ideal of T, since T is Clifford (resp., Boole) regular, then $I^2(I : I^2) = I$ (resp., $I^2 = cI$). If I is not an ideal of T, then as in [9, Theorem 1], it is easy to see that $I = c\phi^{-1}(W)$, where $k \subseteq W \subsetneq K$ is a k-vector space. Since $[K : k] = 2$, then $W = k$ and therefore $I = cR$, as desired. □

The following example shows that Theorem 5.1(2) does not hold in general, and hence nor does the converse of (1).

Example 5.3. Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and field of rational numbers, respectively, and let X and Y be indeterminates over $\mathbb{Q}$. Set $V := \mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, $M := X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, $T := \mathbb{Q}(\sqrt{2}) + M$ and $R := \mathbb{Q} + M$.

Both T and R are one-dimensional local Noetherian domains arising from the DVR V, with $\overline{T} = V$ and $\overline{R} = T$. By Theorem 5.1(2), T is Clifford (actually, Boole) regular, whereas R is not. More specifically, the isomorphy class of the ideal $I := X(\mathbb{Q} + \sqrt{2}\mathbb{Q} + \sqrt{3}\mathbb{Q} + M)$ is not regular in $\mathcal{S}(R)$.

The following is an immediate consequence of Theorem 5.1(2).

Example 5.4. Let n be an integer $\geqslant 1$. Let R be a PVD associated to a non-Noetherian valuation (resp., strongly discrete valuation) domain (V, M) with $\dim(V) = n$ and $[V/M : R/M] = 2$. Then R is an n-dimensional local Clifford (resp., Boole) regular domain that is neither integrally closed nor Noetherian.

Next, we provide new examples of Noetherian Boole (hence Clifford) regular domains (with regard to Example 4.5).

Example 5.5. Let R be a PVD associated to a DVR (V, M) with $[V/M : R/M] = 2$. Then R is a one-dimensional local Noetherian Clifford Boole regular domain that is not integrally closed.

Now, we introduce a useful class of domains that may help constructing more original examples for Clifford or Boole regularity. An integral domain A is said to be *conducive* if the conductor $(A : B)$ is nonzero for each overring B of A other than its quotient field. Examples of conducive domains include pseudo-valuation domains and, in general, arbitrary pullbacks of the form $R := D + M$ arising from a valuation domain $V := K + M$ [18, Propositions 2.1 and 2.2]. Suitable background on conducive domains is [7,18].

We are now able to announce our second theorem of this section. It treats Clifford regularity, for the remaining case "$k = K$", for pullbacks $R := \phi^{-1}(D)$ where D is a conducive domain.

Theorem 5.6. *Under the above notation, suppose that D is a semilocal conducive domain with quotient field $k = K$ and either* (a) *or* (b) *holds*:

(a) *T is a valuation domain,*
(b) *$T := K[X]$ and $R := D + XK[X]$, where X is an indeterminate over K.*

Then R is Clifford regular if and only if so is D.

The proof of (a) is actually handled by the following technical lemma.

Lemma 5.7. *Under the above notation, suppose that T is a valuation domain and for each D-submodule W of K containing D, either W is a ring or $(D : W) \neq 0$. Then R is Clifford regular if and only if so is D.*

Proof. We need only prove the "if" assertions. Assume that D is Clifford regular. Let I be a nonzero (integral) ideal of R. If $M \subsetneqq I$, then $I = \phi^{-1}(J)$ for some nonzero ideal J of D. Since D is Clifford regular, then $J^2(J : J^2) = J$. By [28, Proposition 6], it is easy to see that $I^2(I : I^2) = I$. Assume that $I \subseteq M$. If I is an ideal of T, we are done (since T is Clifford regular). If I is not an ideal of T, then as in [9, Theorem 1], it is easy to see that $I = c\phi^{-1}(W)$, where W is a D-module with $D \subseteq W \subsetneqq K$. If W is a ring, then clearly $W^2(W : W^2) = W$ and therefore $I^2(I : I^2) = I$ by Lemma 5.2. If $(D : W) \neq (0)$, then dW is an (integral) ideal of D for some nonzero element d of D. Since D is Clifford regular, then $(dW)^2(dW : (dW)^2) = dW$ so that $W^2(W : W^2) = W$. Therefore $I^2(I : I^2) = I$ by Lemma 5.2. □

Proof of Theorem 5.6. (a) Follows easily from Lemma 5.7.

(b) Assume that D is Clifford regular. Let I be a nonzero ideal of R. Then $I = f(X)(F + XK[X])$, where F is a nonzero D-submodule of K such that $f(0)F \subseteq D$ [17, Proposition 4.12]. Since D is conducive, then F is a fractional ideal of R. Hence $F^2(F : F^2) = F$ and therefore $I^2(I : I^2) = I$, as desired. □

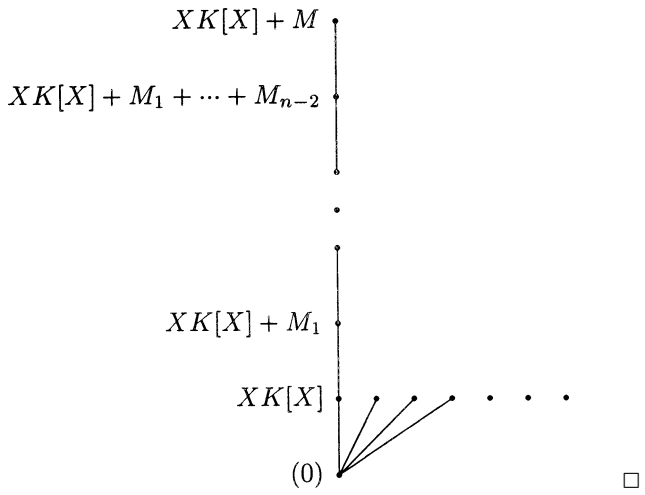
Clearly, Theorems 5.1 and 5.6 generate new families of examples of Clifford regular domains, as shown by the following construction.

Example 5.8. For every positive integer $n \geqslant 2$, there exists an example of an integral domain R satisfying the following conditions:

(1) $\dim(R) = n$,
(2) R is neither integrally closed nor Noetherian,
(3) R is Clifford regular,

(4) each overring of R is Clifford regular,
(5) R has infinitely many maximal ideals.

Proof. Here is an explicit example. Let $n \geqslant 1$ and let $X, X_1, \ldots, X_{n-1}$ be indeterminates over $\mathbb{Q}$. Set $V_1 := \mathbb{Q}(\sqrt{2}) + M_1$, where $M_1 := X_1\mathbb{Q}(\sqrt{2})[X_1]_{(X_1)}$; $V_i := V_{i-1} + M_i$, where $M_i := X_i\mathbb{Q}(\sqrt{2}X_1, \ldots, X_{i-1})[X_i]_{(X_i)}$ for each $2 \leqslant i \leqslant n-1$; $M := M_1 + \cdots + M_{n-1}$; $D := \mathbb{Q} + M$; $K = \mathbb{Q}(\sqrt{2}, X_1, \ldots, X_{n-1})$; and $R := D + XK[X]$. Clearly, $V := V_{n-1} = \mathbb{Q}(\sqrt{2}) + M$ is an $(n-1)$-dimensional valuation domain with maximal ideal M [9, Theorem 2.1], $\overline{R} := V + XK[X]$, and hence R is an n-dimensional non-integrally closed non-Noetherian domain [2,9,16,17,26]. Further, R is Clifford regular by Theorems 5.1 and 5.6. Now let S be an overring of R. Since $V \in \overline{S}$ and $\mathrm{qf}(D) = \mathrm{qf}(V) = K$, it easily can be seen that $V \subseteq S$, hence $\overline{R} \subseteq S$. Consequently, S is Clifford regular since $\overline{R}$ is. Finally, $\mathrm{Spec}(R)$ has the following shape [2,9,17]:



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