1. Evaluate the limit if it exists.

(a) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$. $\frac{0}{0}$, undefined. (4 points)

By Factoring, we get

$$\lim_{x\to 4} \frac{x(x-4)}{(x+1)(x-4)} \qquad \boxed{2 \text{ pts}}$$

$$= \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{5} \qquad \boxed{2 \text{ pts}}$$

(b) $\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{2}}{1-x}$. $\frac{0}{0}$, undefined (6 points)

We multiply by the Conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{2}}{1-x} \cdot \frac{\sqrt{x^2+1}+\sqrt{2}}{\sqrt{x^2+1}+\sqrt{2}} = \lim_{x\to 1} \frac{x^2-1}{(1-x)(\sqrt{x^2+1}+\sqrt{2})} \quad 2$$

$$= \lim_{x\to 1} \frac{-(x+1)}{\sqrt{x^2+1}+\sqrt{2}} \quad 2 \quad = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} \quad 2$$

(c) $\lim_{x\to 0^+} \frac{3}{x}\left(\frac{1}{4+x} - \frac{1}{4-x}\right)$. $+\infty \cdot 0$; undefined (6 points)

$$= \lim_{x\to 0^+} \frac{3}{x}\left(\frac{(4-x)-(4+x)}{(4+x)(4-x)}\right) = \lim_{x\to 0^+} \frac{3}{x}\left(\frac{-2x}{(4+x)(4-x)}\right)$$

$$= \lim_{x\to 0^+} \frac{-6}{(4+x)(4-x)} = \frac{-6}{16} = \frac{-3}{8} \quad 2$$

4

(d) $\lim_{x\to 2^-} ([x-1]-x^2)$, where $[\cdot]$ denotes the greatest integer function. (3 points)

2 $\lim_{x\to 2^-} [x-1] = 0$ (as $x \to 2^-$, then $x<2$)

1 So $\lim_{x\to 2^-} ([x-1]-x^2) = 0-4 = -4$

(e) $\lim_{x \to +\infty} \frac{\cos^2 x}{x^3}$. (6 points)

Since $x \to +\infty$, then $x > 0$.

$$\left.\begin{array}{l} 0 \le \cos^2 x \le 1 \\ \Rightarrow \quad 0 \le \frac{\cos^2 x}{x^3} \le \frac{1}{x^3} \quad (\text{as } x > 0) \end{array}\right\} \begin{array}{l} 1 \\ 1 \end{array}$$

Since $\lim_{x \to +\infty} 0 = 0$ & $\lim_{x \to +\infty} \frac{1}{x^3} = 0$, $\}$ 1 + 1

then by the Squeeze Theorem

$$\lim_{x \to +\infty} \frac{\cos^2 x}{x^3} = 0$$

1
1

(f) $\lim_{x \to -\infty} (x + \sqrt{x^2 + 2x})$. $-\infty + \infty$, undefined (8 points)

$$\lim_{x \to -\infty} x + \sqrt{x^2+2x} \cdot \frac{x - \sqrt{x^2+2x}}{x - \sqrt{x^2+2x}} \quad 1$$

$$= \lim_{x \to -\infty} \frac{-2x}{x - \sqrt{x^2+2x}} \quad 1$$

$$= \lim_{x \to -\infty} \frac{-2x}{x - |x|\sqrt{1 + \frac{2}{x}}} \quad 2$$

$$= \lim_{x \to -\infty} \frac{-2x}{x + x\sqrt{1 + \frac{2}{x}}} \quad 2 \quad (\text{as } x \to -\infty,\ |x| = -x)$$

$$= \lim_{x \to -\infty} \frac{-2}{1 + \sqrt{1 + \frac{2}{x}}} \quad 1$$

$$= \frac{-2}{1+1} = -1 \quad 1$$

2. Use the graph of $f(x) = \dfrac{1}{x}$ to find a number δ such that

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \frac{1}{5} \quad \text{whenever} \quad |x-3| < \delta.$$

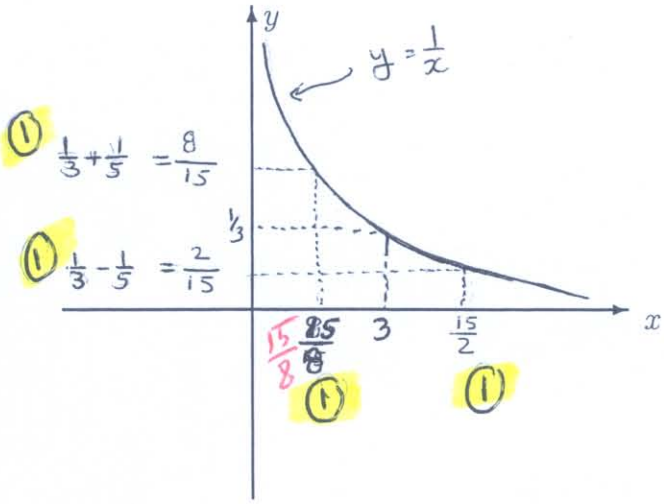
(7 points)

$\delta = \text{minimum}\left(3 - \frac{\cancel{15}}{8}, \frac{15}{2} - 3\right)$ 2

$= \text{minimum}\left(\frac{9}{8}, \frac{9}{2}\right)$

$= \frac{9}{8}$ 1

(or any smaller positive number)



3. Consider the function

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ e - \ln x & \text{if } x > 1 \end{cases}.$$

(a) Is f continuous from the left at 0. Justify. (6 points)

We check if $\lim_{x \to 0^-} f(x) = f(0)$.

- $f(0) = e^0 = 1$ 2
- $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = 2$ 2

Since $\lim_{x\to 0^-} f(x) \neq f(0)$, then f is not continuous from the left at 0. 2

(b) Is f continuous at 1. Justify. (6 points)

We check if $\lim_{x\to 1} f(x) = f(1)$

1 • $f(1) = e^1 = e$

1+1+1 $\left\{\begin{array}{l} \bullet \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^x = e \\ \bullet \lim_{x\to 1^+} f(x) = \lim_{x \to 1^+} (e - \ln x) = e - 0 = e \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = e$

2 Since $\lim_{x\to 1} f(x) = f(1)$, then f is continuous at 1.

4. Where is the function $f(x) = \dfrac{\sin\left(\frac{1}{x}\right)}{e^x - 2}$ continuous. (6 points)

f is continuous in its domain

- $\sin(\frac{1}{x})$ is undefined when $x=0$ (2)
- $e^x - 2 = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$ (2)

Thus f is continuous at every point in

$(-\infty, 0) \cup (0, \ln 2) \cup (\ln 2, +\infty)$ (2)

(OR, f is conts when $x \neq 0, \ln 2$)

5. Show that the equation $e^x = -1 - 2x$ has a root in the interval $(-1, 0)$. (8 points)

We apply the Intermediate Value Theorem by letting

(2) $f(x) = e^x + 2x + 1$, $[-1, 0]$, $N = 0$.

(2) - f is continuous on $[-1, 0]$

(1) - $f(-1) = e^{-1} - 2 + 1 = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$

(1) $f(0) = e^0 + 0 + 1 = 2 > 0$

So $N = 0$ is between $f(-1)$ & $f(0)$

By The Intermediate Value Theorem, there is a number c in $(-1, 0)$ such that

$$f(c) = 0,$$

(2) that is, $e^c + 2c + 1 = 0$

or $e^c = -1 - 2c$.

Thus the given equation has a root in the interval $(-1, 0)$.

6. Find the horizontal and vertical asymptotes of the graph of the function $f(x) = \dfrac{4x-1}{x^3-8x^2}$. Explain. (9 points)

Horizontal Asymptotes

$$\lim_{x\to+\infty} \frac{4x-1}{x^3-8x^2} = \lim_{x\to+\infty} \frac{\frac{4}{x^2}-\frac{1}{x^3}}{1-\frac{8}{x}} \quad (1)$$

$$= \frac{0}{1} = 0$$

$$\lim_{x\to-\infty} \frac{4x-1}{x^3-8x^2} = \lim_{x\to-\infty} \frac{\frac{4}{x^2}-\frac{1}{x^3}}{1-\frac{8}{x}} \quad (1)$$

$$= \frac{0}{1} = 0$$

Thus f has one H.A. : $y=0$ (1)

Vertical Asymptotes

$$f(x) = \frac{4x-1}{x^3-8x^2} = \frac{4x-1}{x^2(x-8)}$$

$x^2(x-8)=0 \Rightarrow x=0,\ x=8$.

- Since $\lim_{x\to 0} \dfrac{4x-1}{x^2(x-8)} = +\infty$, then (2)

$x=0$ is a V.A. (1)

- Since $\lim_{x\to 8^+} \dfrac{4x-1}{x^2(x-8)} = +\infty$ (2)

(or $\lim_{x\to 8^-} \dfrac{4x-1}{x^2(x-8)} = -\infty$),

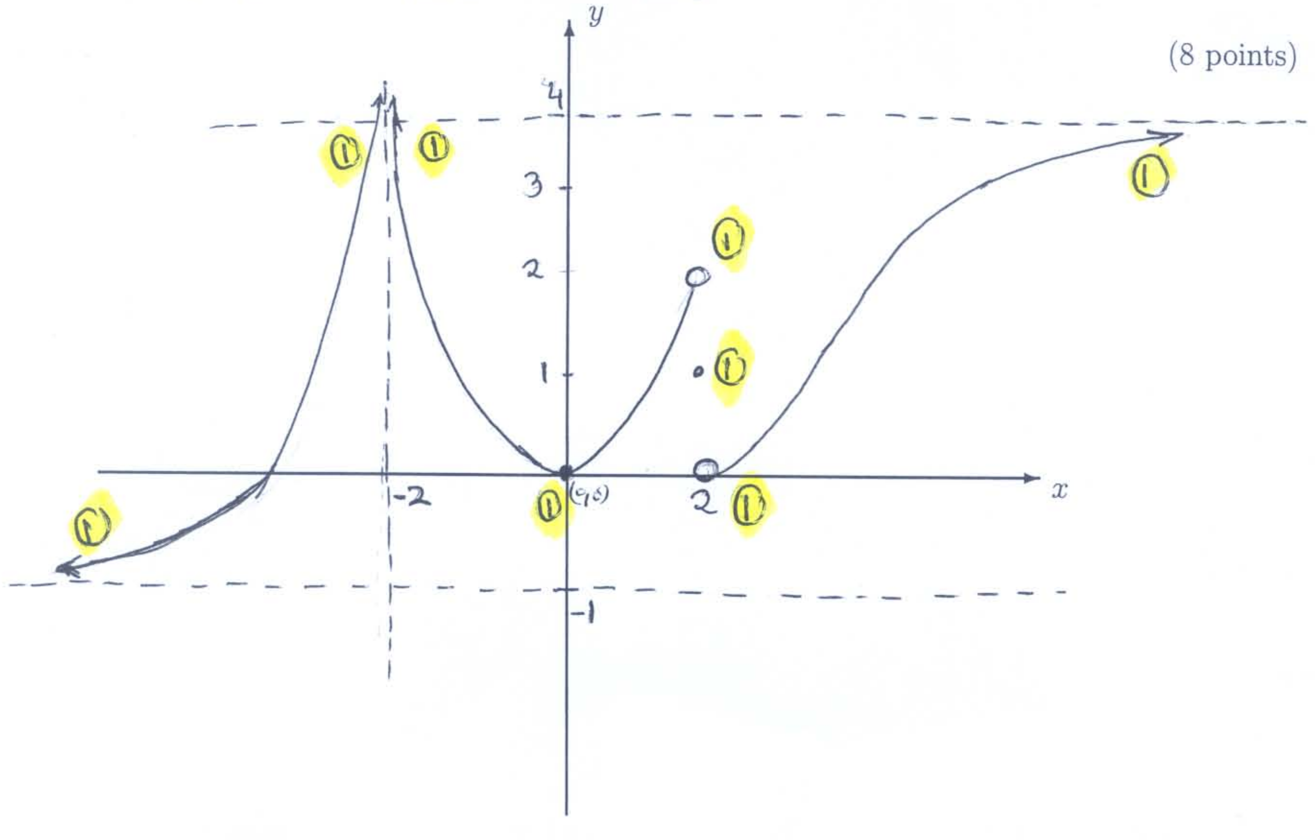
then $x=8$ is a V.A. (1)

Thus f has two V.A.: $x=0$ & $x=8$

7. Sketch the graph of a function f that satisfies all of the given conditions:

$$f(0)=0,\ f(2)=1,\ \lim_{x\to 2^+} f(x) = 0,\ \lim_{x\to 2^-} f(x) = 2,$$
$$\lim_{x\to+\infty} f(x) = 4,\ \lim_{x\to-\infty} f(x) = -1,\ \lim_{x\to -2} f(x) = +\infty.$$

(8 points)



8. Find an equation of the tangent line to the curve $y = \dfrac{1}{x-2}$ at the point $\left(4, \dfrac{1}{2}\right)$.
[**You must use limits**] (8 points)

Slope : $m = \lim_{x \to 4} \dfrac{f(x) - f(4)}{x-4}$ ①

$= \lim_{x \to 4} \dfrac{\frac{1}{x-2} - \frac{1}{2}}{x-4}$

$= \lim_{x \to 4} \dfrac{\frac{4-x}{2(x-2)}}{x-4} = \lim_{x \to 4} \dfrac{-1}{2(x-2)}$ ②

$= \dfrac{-1}{4}$ ②

The equation of the tangent line is

$y - \frac{1}{2} = -\frac{1}{4}(x - 4)$ ②

$\Rightarrow \quad y = -\frac{1}{4}x + \frac{3}{2}$ ①

9. The position function of a particle moving in a straight line is given by the equation of motion $s = t^3 - 2t$, where t is measured in seconds and s in meters.

(a) Find the average velocity of the particle over the time interval $[1, 3]$. (3 points)

average velocity $= \dfrac{S(3) - S(1)}{3-1}$ ②

$= \dfrac{21 - (-1)}{2} = 11$ m/s. ①

(b) Use limits to find the instantaneous velocity of the particle when $t = 2$. (6 points)

$v(a) = \lim_{h \to 0} \dfrac{S(2+h) - S(2)}{h}$ ②

$= \lim_{h \to 0} \dfrac{(2+h)^3 - 2(2+h) - 4}{h}$

$= \lim_{h \to 0} \dfrac{8 + 12h + 6h^2 + h^3 - 4 - 2h - 4}{h}$

$= \lim_{h \to 0} \dfrac{h^3 + 6h^2 + 10h}{h}$ ②

$= \lim_{h \to 0} h^2 + 6h + 10$

$= 10$ m/s ②