*SOLUTIONS- Math 102 - 051 * 2nd Exam

Question One (12-Points)

a. Use cylindrical method to find the solid when the region enclosed by the function $y = 1 - x^2$ and the x − *axis*, is revolved about the x − *axis*, where $-1 \le x \le 1$. (Do not evaluate the integral). **(5-points, suggested time =10 min.)**

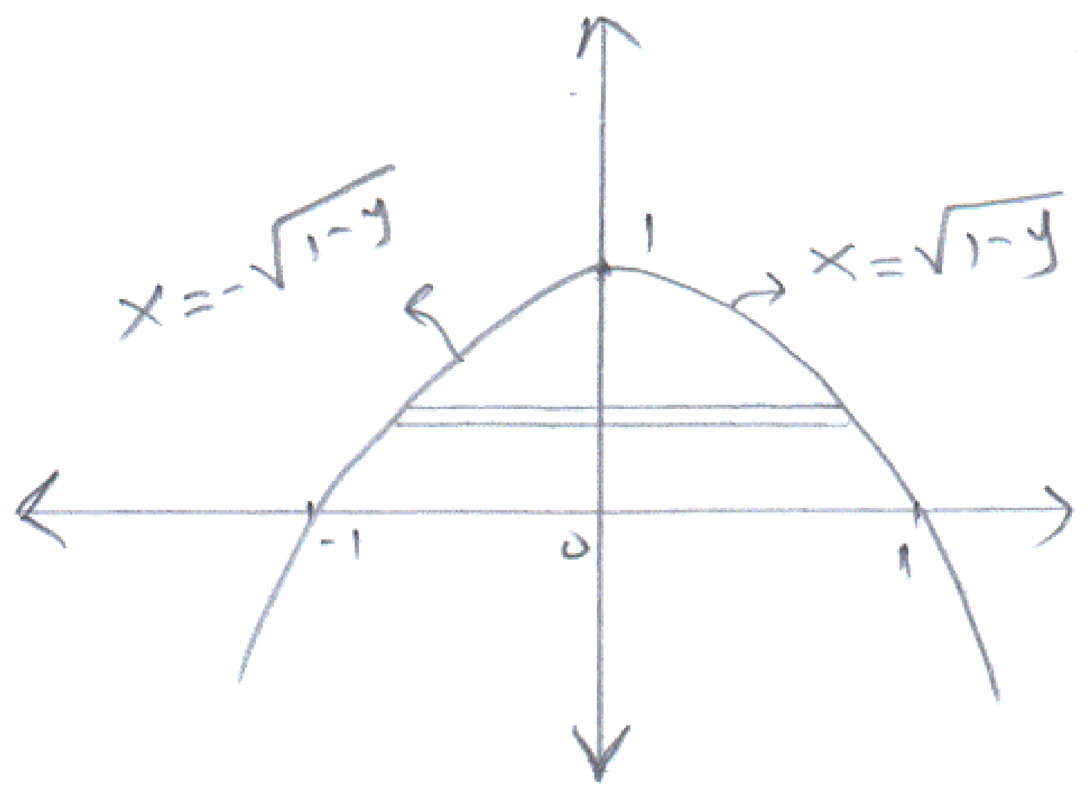
$x^2 = 1 - y \Rightarrow x = \pm\sqrt{1-y}$

$\pm\sqrt{1-y} = 0 \Rightarrow y = 1$, and $y = 0 \;\}\; (1)$

$V = 2\pi \int_0^1 y\left(\sqrt{1-y} - (-\sqrt{1-y})\right) dy \;\}\; (2)$

$= 2\pi \int_0^1 y\left(2\sqrt{1-y}\right) dy$

$= 4\pi \int_0^1 y\sqrt{1-y}\, dy \;\}\; (2)$



b. Use the washers method to find the volume of the solid generated by revolving the region enclosed by $y = \sqrt{4x}$, $y = 2x$ about the line $y = -1$ **(7-points, suggested time =10 min.)**

$\sqrt{4x} = 2x \Rightarrow 4x = 4x^2$

$4x - 4x^2 = 0 \Rightarrow 4x(1-x) = 0 \Rightarrow x = 0, x = 1 \;\}\; (2)$

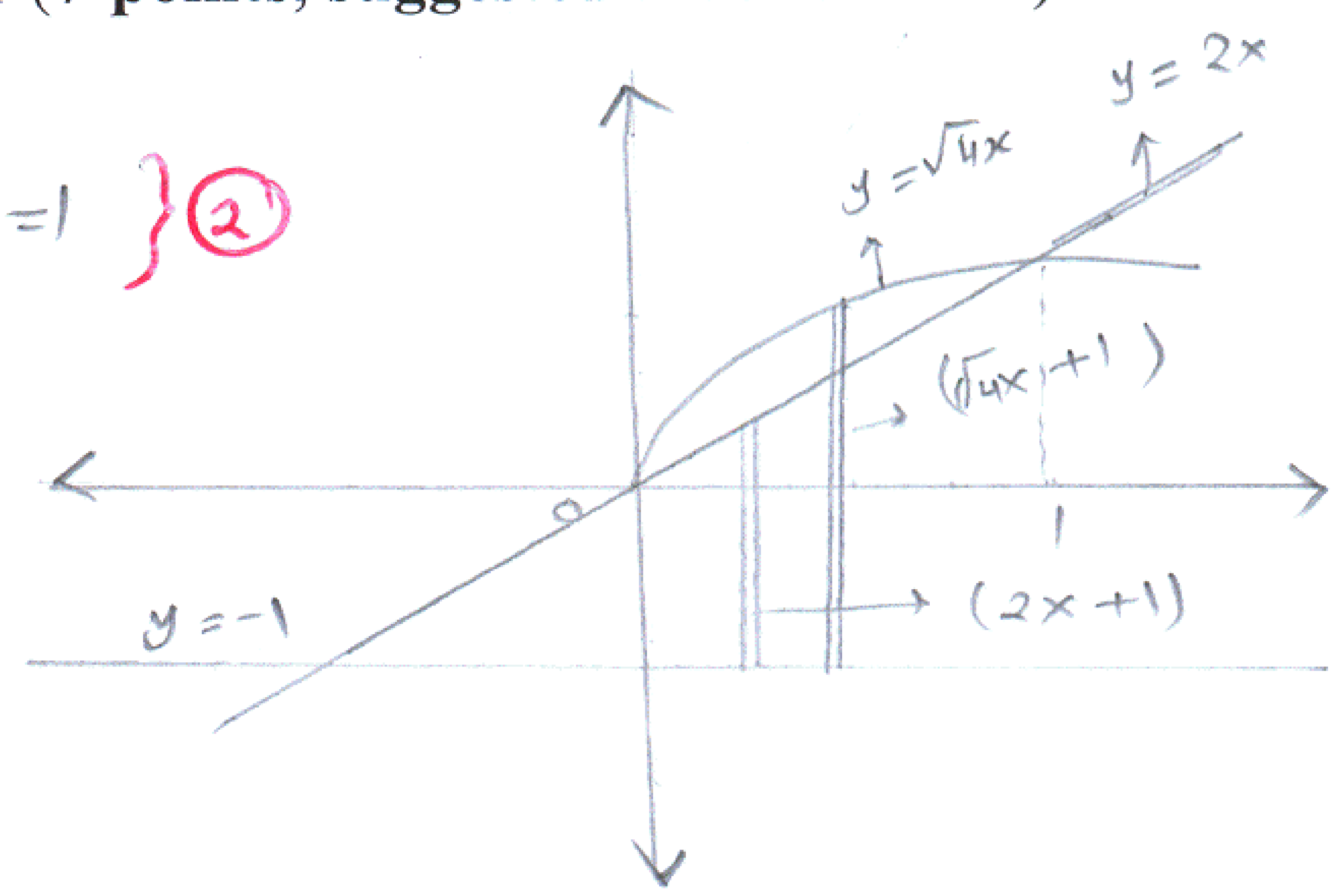
$V = \pi \int_0^1 \left[(\sqrt{4x}+1)^2 - (2x+1)^2\right] dx \;\}\; (2)$

$= \pi \int_0^1 (\cancel{4x} + 2\sqrt{4x} + \cancel{1} - 4x^2 - \cancel{4x} - \cancel{1}) \;\}\; (1)$

$= \pi \int_0^1 (4x^{1/2} - 4x^2)\, dx$

$= \pi \left[\frac{8}{3} x^{3/2} - \frac{4}{3} x^3 \right]_0^1 \;\}\; (1)$

$= \pi \left(\frac{8}{3} - \frac{4}{3} - 0 \right) = \frac{4\pi}{3} \;\}\; (1)$



Question Two (14-Points) * SOLUTIONS – Math 102-051 * 2nd-Exam

a. Find the arc length of the curve $x = \frac{1}{24}y^3 + \frac{2}{y}$ from $y = 2$ to $y = 4$ (Do not evaluate the integral). **(6-points, suggested time =6 min)**

$$\frac{dx}{dy} = \frac{3}{24}y^2 - \frac{2}{y^2} = \frac{1}{8}y^2 - 2y^{-2} \quad \Big\} \; 2$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{8}y^2 - 2y^{-2}\right)^2 = 1 + \frac{1}{64}y^4 - \frac{1}{2} + 4y^{-4}$$

$$= \frac{1}{64}y^4 + \frac{1}{2} + 4y^{-4} = \left(\frac{1}{8}y^2 + 2y^{-2}\right)^2 \quad \Big\} \; 2$$

$$L = \int_2^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_2^4 \left(\frac{1}{8}y^2 + 2y^{-2}\right) dy \quad \Big\} \; 2$$

b. Find the area of the surface generated by revolving the parametric curve given by the equations: $x(t) = \frac{3}{\sqrt{\pi}}\sin^2(t), y(t) = \frac{4}{\sqrt{\pi}}\cos^2(t); 0 \le t \le \frac{\pi}{2}$ about the $x-axis$.

(8-points, suggested time =10 min.)

$$x'(t) = \frac{3}{\sqrt{\pi}} \cdot 2\sin t\cos t = \frac{3}{\sqrt{\pi}}\sin 2t \quad \Big\} \; 1$$

$$y'(t) = \frac{-4}{\sqrt{\pi}} 2\cos t\sin t = \frac{-4}{\sqrt{\pi}}\sin 2t \quad \Big\} \; 1$$

$$[x'(t)]^2 + [y'(t)]^2 = \frac{9}{\pi}\sin^2 2t + \frac{16}{\pi}\sin^2 2t = \frac{25}{\pi}\sin^2 2t \quad \Big\} \; 2$$

$$S = 2\pi \int_0^{\frac{\pi}{2}} y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{4}{\sqrt{\pi}}\cos^2 t \cdot \sqrt{\frac{25}{\pi}\sin^2 2t}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{4}{\sqrt{\pi}}\cos^2 t \cdot \frac{5}{\sqrt{\pi}}\sin 2t\, dt, \quad \sin 2t = 2\sin t\cos t$$

$$= 80 \int_0^{\frac{\pi}{2}} \cos^3 t \cdot \sin t\, dt, \quad u = \cos t \rightarrow du = -\sin t\, dt \quad \Big\} \; 3$$

$t = 0 \rightarrow u = 1$

$t = \frac{\pi}{2} \rightarrow u = 0$

$$= -80 \int_1^0 u^3 \cdot du = -80 \cdot \frac{u^4}{4}\Big]_1^0 = -20(0-1) = 20 \quad \Big\} \; 1$$

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Question Three (8 -Points)
Answer the following as TRUE or FALSE: **(2-points each, suggested time =10 min.)**

a. $\sec h\left(\ln\left(x\right)\right)=\dfrac{x^2+1}{2x}$:---False---- 2

$$\operatorname{sech}(\ln(x)) = \frac{2}{e^{\ln x}+e^{-\ln x}} = \frac{2}{x+\frac{1}{x}} = \frac{2}{\frac{x^2+1}{x}} = \frac{2x}{x^2+1}$$

b. $\tanh\left(-\ln(2)\right)=-\dfrac{3}{17}$:---False---- 2

$$\tanh(-\ln 2) = \frac{e^{-\ln 2}-e^{+\ln 2}}{e^{-\ln 2}+e^{\ln 2}} = \frac{\frac{1}{2}-2}{\frac{1}{2}+2} = \frac{\frac{1-4}{2}}{\frac{1+4}{2}} = \frac{-3}{5}$$

c. $\dfrac{d}{dx}(\sinh(\ln(x))=\dfrac{x^2+1}{2x^2}$:---True---- 2

$$\frac{d}{dx}(\sinh(\ln(x)) = \frac{1}{x}\cosh(\ln x) = \frac{1}{x}\cdot\frac{e^{\ln x}+e^{-\ln x}}{2} = \frac{1}{x}\cdot\frac{x+\frac{1}{x}}{2}$$

$$= \frac{1}{x}\cdot\frac{\frac{x^2+1}{x}}{2} = \frac{1}{x}\cdot\frac{x^2+1}{x}\cdot\frac{1}{2} = \frac{x^2+1}{2x^2}$$

d. If $\sinh(k)=2$, then $\cosh(k)=\pm\sqrt{5}$:---False---- 2

$$\cosh^2 k - \sinh^2 k = 1$$

$$\cosh^2 k = 1 + \sinh^2 k = 1+4 = 5$$

$$\Rightarrow \cosh k = +\sqrt{5}$$

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Question Four (25-Points)

Evaluate the following integrals:

a. $\int \tan^2(x)\sec^6(x)\,dx$ **(4-points, suggested time =5 min.)**

$$\int \tan^2 x \sec^6 x\,dx = \int \tan^2 x \cdot \sec^2 x \cdot \sec^2 x \cdot \sec^2 x\,dx \quad , \sec^2 x = 1+\tan^2 x$$

$$= \int \tan^2 x\,(1+\tan^2 x)^2 \cdot \sec^2 x\,dx$$

$$= \int \tan^2 x\,(1+2\tan^2 x+\tan^4 x)\sec^2 x\,dx \Big\}\ (1)$$

let $u = \tan x$, $du = \sec^2 x\,dx$ } (1)

$$= \int u^2(1+2u^2+u^4)\,du$$

$$= \int (u^2+2u^4+u^6)\,du$$

$$= \frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7 + C \Big\}\ (1)$$

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C \Big\}\ (1)$$

b. $\int 2x \sin^{-1}\left(\frac{2}{x}\right)dx$ **(6-points, suggested time =8 min.)**

Let $u = \sin^{-1}\left(\frac{2}{x}\right)$ $\qquad dv = 2x\,dx$

$du = -\frac{2}{x^2}\cdot\frac{1}{\sqrt{1-\frac{4}{x^2}}}\,dx$ (2) $\qquad v = x^2$ (1)

$= -\frac{2}{x^2}\cdot\frac{x}{\sqrt{x^2-4}}\,dx$ $\qquad v = x^2$

$$\int 2x\sin^{-1}\left(\frac{2}{x}\right)dx = x^2\sin^{-1}\left(\frac{2}{x}\right) + \int \frac{2x}{\sqrt{x^2-4}}\,dx$$

, put $y = x^2-4$, $dy = 2x\,dx$ } (1)

$$= x^2\sin^{-1}\left(\frac{2}{x}\right) + \int y^{-1/2}\,dy$$

$$= x^2\sin^{-1}\left(\frac{2}{x}\right) + 2y^{1/2} + C \Big\}\ (1)$$

$$= x^2\sin^{-1}\left(\frac{2}{x}\right) + 2\sqrt{x^2-4} + C \Big\}\ (1)$$

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c. $\int \frac{dx}{\sin(x)+\cos(x)}$ **(5-points, suggested time =5 min)**

Let $u = \tan(\frac{x}{2}) \Rightarrow \sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$

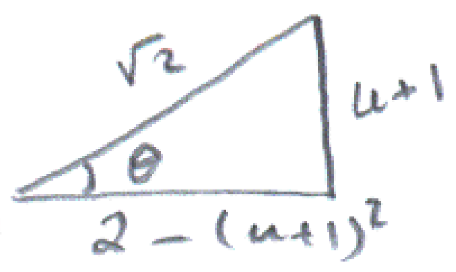
$dx = \frac{2}{1+u^2} du$ $\Big\}$ 3

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{\frac{2}{1+u^2} du}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} = \int \frac{\frac{2}{\cancel{1+u^2}}}{\frac{2u+1-u^2}{\cancel{1+u^2}}} du$$

1 $$= -\int \frac{2}{u^2-2u-1} du$$

$u^2-2u-1 = u^2-2u+1-1-1$
$= (u+1)^2 - 2$

$$= -\int \frac{2}{(u+1)^2-2} du \quad , \quad u+1 = \sqrt{2}\sin\theta, \quad du = \sqrt{2}\cos\theta \, d\theta$$

1 

$$= -2\int \frac{\sqrt{2}\cos\theta \, d\theta}{2\sin^2\theta - 2} = \frac{-2\cdot\sqrt{2}}{-2}\int \frac{\cos\theta}{1-\sin^2\theta} d\theta = \sqrt{2}\int \frac{1}{\cos\theta} d\theta$$

$$= \sqrt{2}\int \sec\theta \, d\theta = \sqrt{2}\ln|\sec\theta + \tan\theta| + c$$

$$= \sqrt{2}\ln\left|\frac{\sqrt{2}}{2-(u+1)^2} + \frac{u+1}{2-(u+1)^2}\right| + c = \sqrt{2}\ln\left|\frac{\tan\frac{x}{2} + \sqrt{2} + 1}{2-(\tan\frac{x}{2}+1)^2}\right| + c$$

d. $\int \frac{dx}{4x\,(\sqrt{x} - \sqrt[4]{x})}$ **(10-points, suggested time =10 min.)**

Let $u = x^{1/4} \Rightarrow u^4 = x \Rightarrow 4u^3 du = dx$ $\Big\}$ 1

$$I = \int \frac{dx}{4x(x^{1/2} - x^{1/4})} = \int \frac{4u^3 du}{4u^4(u^2-u)} = \int \frac{du}{u(u^2-u)} = \int \frac{du}{u^2(u-1)} . \Big\} \ 2$$

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} = \frac{Au(u-1) + B(u-1) + Cu^2}{u^2(u-1)} \Big\} \ 2$$

$Au(u-1) + B(u-1) + Cu^2 = 1$

$u = 0 \Rightarrow \quad 0 + (-B) + 0 = 1 \Rightarrow B = -1$ $\Big\}$ 1

$u = 1 \Rightarrow \quad 0 + 0 + C = 1 \Rightarrow C = 1$ $\Big\}$ 1

$u = 2 \Rightarrow \quad 2A + B + 4C = 1 \Rightarrow 2A = 1 - (-1) - 4(1) = -2 \Rightarrow A = -1$ $\Big\}$ 1

$$I = \int \left(\frac{-1}{u} - \frac{1}{u^2} + \frac{1}{u-1} \right) du = \int \left(\frac{1}{u-1} - \frac{1}{u} - u^{-2} \right) du$$

$\Big\}$ 1

$$= \ln|u-1| - \ln|u| - \frac{u^{-1}}{-1} + c$$

$$= \ln\left|\frac{u-1}{u}\right| + \frac{1}{u} + c = \ln\left|\frac{x^{1/4}-1}{x^{1/4}}\right| + \frac{1}{x^{1/4}} + c . \Big\} \ 1$$

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Question Five (11-Points)

Consider the improper integral: $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

a. Why it is an improper integral? **(2-points, suggested time =1 min.)**

1. Infinite interval $[1, \infty)$
2. Infinite discontinuity point at $x = 1$ } (2)

b. Evaluate the integral if it converges. **(9-points, suggested time =10 min.)**

$$\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \underbrace{\int_1^{2} \frac{dx}{x\sqrt{x^2-1}}}_{I_1} + \underbrace{\int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}}}_{I_2} \quad \} \ (1)$$

(1)

$$I_1 = \lim_{R \to 1^+} \int_R^2 \frac{dx}{x\sqrt{x^2-1}}, \quad \int \frac{dx}{x\sqrt{x^2-1}} \qquad \text{let } x = \sec\theta$$

$$dx = \sec\theta \tan\theta \, d\theta$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta \tan\theta}{\sec\theta \sqrt{\sec^2\theta - 1}} d\theta = \int \frac{\cancel{\sec\theta} \tan\theta}{\cancel{\sec\theta} \cdot \sqrt{\tan^2\theta}} d\theta$$

$$= \int \frac{\tan\theta}{\tan\theta} d\theta = \int d\theta = \theta + C = \sec^{-1}(x) + C \quad \} \ (3)$$

$$\therefore I_1 = \lim_{R \to 1^+} \sec^{-1}(x) \Big]_R^2 = \lim_{R \to 1^+} \left(\sec^{-1}(2) - \sec^{-1}(R) \right)$$

$$= \sec^{-1}(2) - 0 = \sec^{-1}(2) \quad \} \ (1)$$

$$I_2 = \lim_{l \to \infty} \int_2^{l} \frac{dx}{x\sqrt{x^2-1}} = \lim_{l \to \infty} \sec^{-1}(x) \Big]_2^{l}$$

(1)

$$= \lim_{l \to \infty} \left(\sec^{-1}(l) - \sec^{-1}(2) \right)$$

$$= \frac{\pi}{2} - \sec^{-1}(2) \quad \} \ (1)$$

$$\therefore \int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(2) + \frac{\pi}{2} - \sec^{-1}(2) = \frac{\pi}{2}, \quad \text{Converges.} \quad \} \ (1)$$