Question One (6-Points) Math 102 – Major One Solutions – 051

Use the definition of the definite integral with x_k^* as the right end- point of each subinterval to find the area under the curve $y = f(x) = x^3$, over the interval $[0,3]$

(Note: all subintervals with the same length, and $A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$)

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$
$$x_k^* = x_k = 0 + \frac{3}{n}k = \frac{3}{n}k$$
② points

$$A_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{n} f\left(\frac{3}{n}k\right)\cdot\frac{3}{n}$$
$$= \frac{3}{n}\sum_{k=1}^{n} \frac{27}{n^3}k^3 = \frac{81}{n^4}\sum_{k=1}^{n} k^3$$
② points

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} 81\cdot\frac{1}{n^4}\sum_{k=1}^{n} k^3$$

$$= 81\cdot\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n} k^3$$
① points.

$$= 81\cdot\frac{1}{4} = \frac{81}{4}.$$
①

Question Two (6-Points)

Answer the following two parts (3-Points each)

a. Express $\sum_{k=4}^{n+2} \frac{e^{-k}}{k^2-2}$ in sigma notation with $k=1$ as a lower limit

$$\sum_{k=4}^{n+2} \frac{e^{-k}}{k^2-2} = \sum_{k=1}^{n-1} \frac{e^{-(k+3)}}{(k+3)^2-2}$$
③ points

b. Express the following sum in closed form: $\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)\frac{1}{n}$

$$\sum_{k=1}^{n}\left(1+\frac{2}{n}k\right)\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{n} + \sum_{k=1}^{n}\frac{2}{n^2}k$$
①

$$= \frac{1}{n}\cdot\sum_{k=1}^{n} 1 + \frac{2}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{1}{\cancel{n}}\cdot\cancel{n} + \frac{\cancel{2}}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}$$
①

$$= 1 + \frac{1}{n}(n+1) = 1 + 1 + \frac{1}{n} = 2 + \frac{1}{n}.$$
①

Question Three (12-Points) Maj. 1 – Math 102 – Solutions – 051

a. Find $f'(3)$ given that $f(x) = \frac{1}{x}\int_3^x (2t - 3f'(t))dt$ (3 -Points)

$$f'(x) = \frac{-1}{x^2}\cdot\int_3^x (2t - 3f'(t))\,dt + \frac{1}{x}\cdot(2x - 3f'(x)) \quad 1$$

$$f'(3) = \frac{-1}{9}\int_3^3 (2t - 3f'(t))\,dt + \frac{1}{3}(6 - 3f'(3)) \quad 1$$

$$= 0 + 2 - f'(3) \Rightarrow 2f'(3) = 2 \Rightarrow f'(3) = 1 \quad 1$$

b. Consider the function $f(x) = \frac{1}{x^2}$; $[1,4]$, then:(3+2=5-Points)

I. Find x^* in the interval that satisfies the Mean Value Theorem for Integrals.

$$1 \rightarrow \int_1^4 f(x)\,dx = f(x^*)(4-1) \Rightarrow \int_1^4 \frac{1}{x^2}\,dx = \frac{1}{x^{*2}}(3)$$

$$\frac{-1}{x}\Big|_1^4 = \frac{3}{x^{*2}} \Rightarrow \frac{3}{4} = \frac{3}{x^{*2}} \quad 1 \Rightarrow x^{*2} = 4$$

$$\Rightarrow x^* = 2 \quad 1 \quad \text{and ignor } x^* = -2 \notin [1,4]$$

II. Find the average value of the function over the interval.

$$f_{ave.} = \frac{\int_1^4 f(x)\,dx}{4-1} \quad 1 \quad = \frac{3/4}{3} = \frac{3}{4}\cdot\frac{1}{3} = \frac{1}{4}. \quad 1$$

c. Answer the following as TRUE or FALSE :(4 –Points)

I. If f is an odd function on $[-\pi,\pi]$, then $\int_{-\pi}^{\pi}\cos(f(x))dx = 0$. :False 1

II. If $\int_1^3 f(t)dt = 9$, then $\int_1^3 \sqrt{f(t)}\,dt = \sqrt{9} = 3$:False 1

III. $\int_{-2}^{3} f(x)dx + \int_3^{-2} g(x)dx = 0$:False 1

IV. $\frac{d}{dx}\int_1^e \frac{1}{x}dx = 1$:False 1

1 pt For $\underline{\underline{c}}$ for indefinite integrals

<u>Question Four (4+5+5+4=18-Points)</u>

Evaluate the following integrals:

a. $\int (2x+1)e^{(2x^2+2x)}dx$

Let $u = 2x^2+2x \Rightarrow du = (4x+2)\,dx \Rightarrow dx = \dfrac{du}{2(2x+1)}$ } (1)

$\int (2x+1)\, e^{(2x^2+2x)}\, dx = \int \cancel{(2x+1)}\, e^{u} \cdot \dfrac{du}{2\cancel{(2x+1)}}$ } (1)

$= \frac{1}{2} e^{u} + c$ } (1)

$= \frac{1}{2} e^{2x^2+2x} + c$ } (1)

b. $I = \int \dfrac{1}{x^2+2x+5}dx$

$x^2+2x+5 = x^2+2x+1-1+5 = (x+1)^2+4$ (1)

$I = \int \dfrac{dx}{(x+1)^2+4}$,

let $u = x+1 \Rightarrow du = dx$ (1)

$\therefore I = \int \dfrac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + c$ } (2) 1 pt for $\underline{\underline{c}}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$ } (1)

c. $\int x \sin^3 x^2 \cos x^2\, dx$

Let $u = \sin x^2 \Rightarrow du = 2x\cos x^2\, dx \Rightarrow dx = \dfrac{du}{2x\cos x^2}$ } (2)

$\int x \sin^3 x^2 \cos x^2\, dx = \int \cancel{x}\, u^3 \cdot \cancel{\cos x^2} \cdot \dfrac{du}{2\cancel{x}\cancel{\cos x^2}}$

$= \frac{1}{2}\int u^3\, du = \frac{1}{2}\cdot\frac{u^4}{4} + c$ } (2)

$= \frac{1}{8}\sin^4 x^2 + c$ } (1)

d. $\int_0^4 |2-x|\,dx$

$|2-x| = |x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x < 2 \end{cases}$

$\underset{0}{\vert}\overset{2-x}{\text{———}}\underset{2}{\vert}\overset{x-2}{\text{———}}\underset{4}{\vert}$ } (1)

$\int_0^4 |2-x|\,dx = \int_0^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$ } (1)

$= 2x - \frac{x^2}{2}\Big|_0^2 + \frac{x^2}{2} - 2x\Big|_2^4$ } (1)

$= 4-2-0 + 8-8-(2-4)$

$= 2 + 2 = 4$ } (1)

Question Five (6+7=13-Points) Math 102-051- Major 1 Solutions

a. Solve the initial value problem : $\frac{dy}{dx} = \frac{1}{x\sqrt{1-(\ln x)^2}}$, $y(1) = 5$

① $y = \int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$, let $u = \ln x \Rightarrow du = \frac{1}{x}dx \Rightarrow dx = x\,du$

$= \int \frac{\cancel{x}\,du}{\cancel{x}\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + c$ } ②

$y(x) = \sin^{-1}(\ln x) + c$

$y(1) = 5 \Rightarrow y(1) = \sin^{-1}(\ln(1)) + c = 5$ ①

$\sin^{-1}(0) + c = 5 \Rightarrow 0 + c = 5 \Rightarrow c = 5$ ②

$\therefore y(x) = \sin^{-1}(\ln x) + 5$

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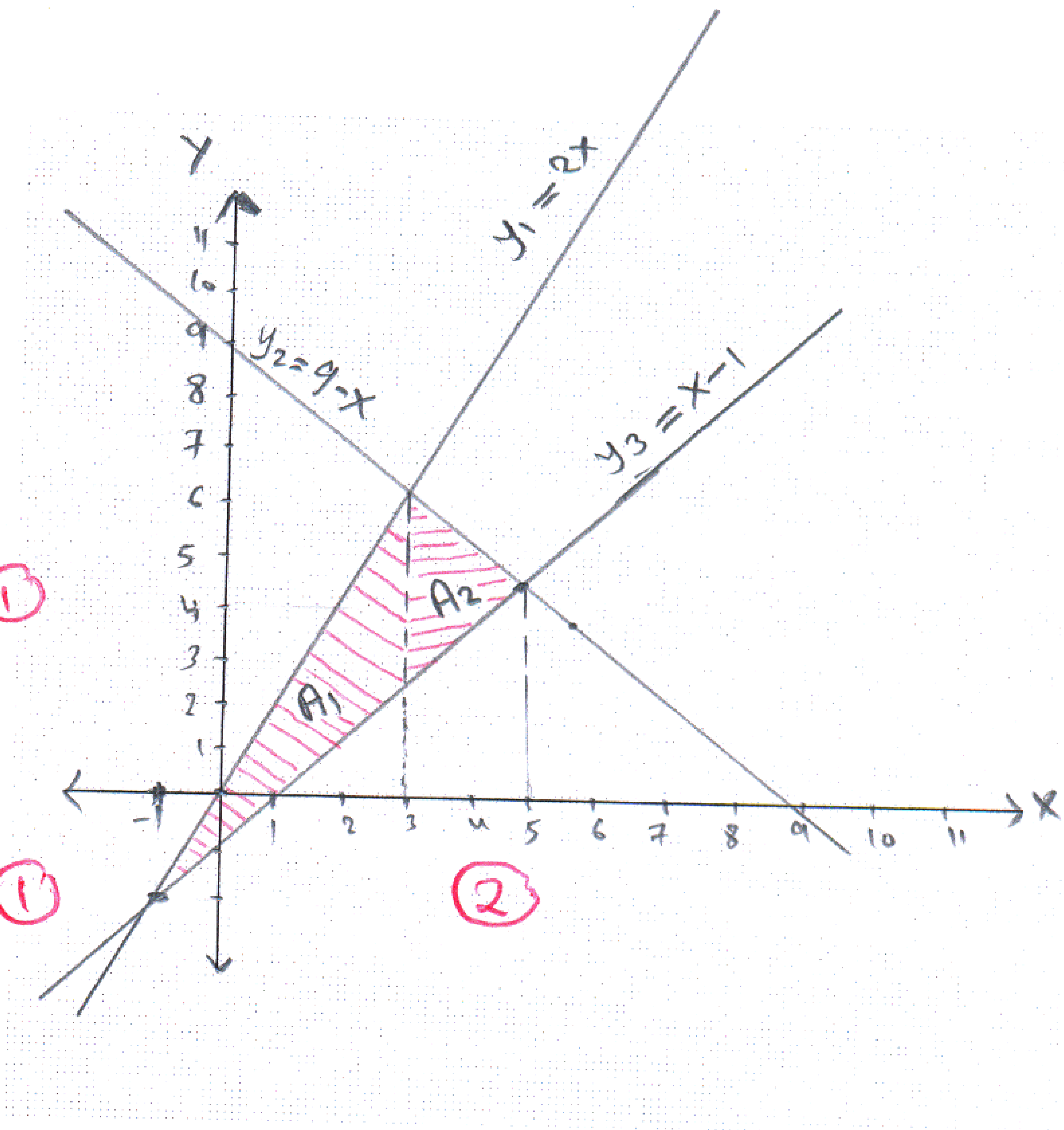
b. Sketch the graph and find the area enclosed by the curves :
$y = 2x$, $x + y = 9$, $y = x - 1$

$y_1 = 2x$, $y_2 = 9 - x$, $y_3 = x - 1$

$y_1 = y_2 \Rightarrow 2x = 9 - x \Rightarrow x = 3$

$y_1 = y_3 \Rightarrow 2x = x - 1 \Rightarrow x = -1$

$y_2 = y_3 \Rightarrow 9 - x = x - 1 \Rightarrow x = 5$



$A = A_1 + A_2$

$A_1 = \int_{-1}^{3} (2x - (x-1))\,dx = \int_{-1}^{3} (x+1)\,dx$ ①

$= \frac{x^2}{2} + x \Big|_{-1}^{3} = 8$ ①

$A_2 = \int_{3}^{5} (9 - x - (x-1))\,dx = \int_{3}^{5} (10 - 2x)\,dx$ ①

$= 10x - x^2 \Big|_{3}^{5} = 4$ ①

$A = 8 + 4 = 12.$ ①