Math 101 – 052
EXAM 1 – Solutions

Question 1

Use the squeezing theorem to find $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$.

For each $x \neq 0$, $-1 \leq \sin\frac{1}{x} \leq 1$.

Then $e^{-1} \leq e^{\sin\frac{1}{x}} \leq e$

Multiplying by x^2, we get: $x^2 e^{-1} \leq x^2 e^{\sin\frac{1}{x}} \leq x^2 e$.

Taking the limit when $x \to 0$, we obtain:

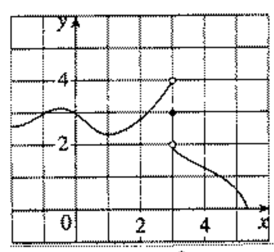
$$\underbrace{\lim_{x\to 0} x^2 e^{-1}}_{0} \leq \lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} \leq \underbrace{\lim_{x\to 0} x^2 e}_{0}.$$

$$0 \leq \lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} \leq 0$$

By Squeezing Theorem, $\lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} = 0$.

Question 2

Let $\lim_{x\to 3} f(x) = 0$ and $\lim_{x\to 3} h(x) = 5$. Use these limits and the given graph of the function g to evaluate each of the following limits if it exists. If the limit does not exist, explain why.



$y = g(x)$

a) $\lim_{x\to 3} \left(f(x) - \frac{h(x)}{3}\right)$

$$= \lim_{x\to 3} f(x) - \frac{1}{3}\lim_{x\to 3} h(x)$$

$$= 0 - \frac{1}{3}\cdot 5 = -\frac{5}{3}$$

b) $\lim_{x\to 3} \frac{g(x)-3}{h(x)}$

- $\lim_{x\to 3^-} \frac{g(x)-3}{h(x)} = \frac{4-3}{5} = \frac{1}{5}$.
- $\lim_{x\to 3^+} \frac{g(x)-3}{h(x)} = \frac{2-3}{5} = -\frac{1}{5}$.

So $\lim_{x\to 3} \frac{g(x)-3}{h(x)}$ does not exist.

c) $\lim_{x\to 3^-} (g(x)+h(x))^3$

$$\Rightarrow \lim_{x\to 3^-} (g(x)+h(x))^3 = \left(\lim_{x\to 3^-} g(x) + \lim_{x\to 3^-} h(x)\right)^3$$

$$= (4+5)^3 = 9^3 = 729.$$

d) $\lim_{x\to 0^+} \frac{1}{\sqrt{g(x)}}$

$$= \frac{1}{\sqrt{\lim_{x\to 0^+} g(x)}} = \frac{1}{\sqrt{3}}$$

e) $\lim_{x\to 3} f(x)g(x)$

- $\lim_{x\to 3^-} f(x)g(x) = \left(\lim_{x\to 3^-} f(x)\right)\left(\lim_{x\to 3^-} g(x)\right) = 0.4 = 0$.
- $\lim_{x\to 3^+} f(x)g(x) = \left(\lim_{x\to 3^+} f(x)\right)\left(\lim_{x\to 3^+} g(x)\right) = 0.2 = 0$.

So $\lim_{x\to 3} f(x)g(x) = 0$

Question 3

By using the ε and δ definition, prove that $\lim\limits_{x\to 4} \dfrac{1}{x-2} = \dfrac{1}{2}$.

- $\left|\dfrac{1}{x-2} - \dfrac{1}{2}\right| < \varepsilon$ whenever $0 < |x-4| < \delta$

$$\frac{|x-4|}{|x-2|} < 2\varepsilon$$

- Assume that $|x-4| < 1$.

$$-1 < x-4 < 1$$

add 2: $1 < x-2 < 3$.

so $1 < |x-2| < 3$.

Then $\dfrac{1}{3} < \dfrac{1}{|x-2|} < 1$. So $\dfrac{|x-4|}{|x-2|} < |x-4| < 2\varepsilon$.

Conclusion, we take $\delta = \text{Min}(1, 2\varepsilon)$.

Question 4

Prove that the equation $x^2 + x - \cos x = 0$ has at least two solutions in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

- Consider the function $f(x) = x^2 + x - \cos x$.

Then $f(x)$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (as a sum of continuous functions)

$$f(-\tfrac{\pi}{2}) = (-\tfrac{\pi}{2})^2 + (-\tfrac{\pi}{2}) - \cos(-\tfrac{\pi}{2}) = \frac{\pi^2}{4} - \frac{\pi}{2} = \frac{\pi^2 - 2\pi}{4} = \frac{\pi(\pi-2)}{4} > 0$$

$$f(\tfrac{\pi}{2}) = (\tfrac{\pi}{2})^2 + (\tfrac{\pi}{2}) - \cos(\tfrac{\pi}{2}) = \frac{\pi^2}{4} + \frac{\pi}{2} = \frac{\pi^2 + 2\pi}{4} > 0.$$

$f(-\frac{\pi}{2}) > 0$ and $f(\frac{\pi}{2}) > 0$, we cannot conclude

① Consider the interval $[-\frac{\pi}{2}, 0]$

$$\left\{\begin{array}{l} f \text{ continuous on } [-\frac{\pi}{2}, 0] \\ f(-\frac{\pi}{2}) = \frac{\pi(\pi-2)}{4} > 0 \\ f(0) = -1 < 0 \end{array}\right\}$$

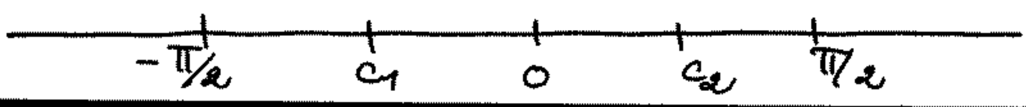
By the Intermediate Value Th. there exists $c_1 \in (-\frac{\pi}{2}, 0)$ such that $f(c_1) = 0$.

② Consider the interval $[0, \frac{\pi}{2}]$

$$\left\{\begin{array}{l} f \text{ continuous on } [0, \frac{\pi}{2}] \\ f(0) = -1 < 0 \\ f(\frac{\pi}{2}) = \frac{\pi^2 + 2\pi}{4} > 0 \end{array}\right\}$$

By The Intermediate value Th. there exists $c_2 \in (0, \frac{\pi}{2})$ such that $f(c_2) = 0$.

Then the equation $x^2 + x - \cos x = 0$ has 2 solutions



Question 5

Suppose that f is a continuous function on the interval $[0,1]$ and $f(0)=f(1)$. Prove $\neq f(\frac{1}{2})$ (analytically and not geometrically) that there exists $a \in (0,\frac{1}{2})$ such that a and $a+\frac{1}{2}$ have the same image, that is, $f(a)=f(a+\frac{1}{2})$.

(**Hint**: consider the function $g(x)=f(x+\frac{1}{2})-f(x)$)

- Consider the function $g(x)=f(x+\frac{1}{2})-f(x)$.

g is continuous on $[0,\frac{1}{2}]$ since f is continuous.

$$g(0)=f(0+\tfrac{1}{2})-f(0)=f(\tfrac{1}{2})-f(0).$$
$$=f(\tfrac{1}{2})-f(1) \text{ since } f(0)=f(1)$$

$$g(\tfrac{1}{2})=f(\tfrac{1}{2}+\tfrac{1}{2})-f(\tfrac{1}{2})$$
$$=f(1)-f(\tfrac{1}{2})=-(f(\tfrac{1}{2})-f(1))$$
$$=-g(0).$$

Then $g(\frac{1}{2})=-g(0)\neq 0$

So $g(1/2)$ and $g(0)$ have opposite signs

(that is, if $g(1/2)>0$, then $g(0)<0$
and if $g(1/2)<0$, then $g(0)>0$)

Then 0 is between $g(0)$ and $g(\frac{1}{2})$.

By the Intermediate Value Theorem, there exists $a\in(0,\frac{1}{2})$ such that $g(a)=0$.

Then $0=g(a)=f(a+\frac{1}{2})-f(a)$.

Therefore, $f(a+\frac{1}{2})=f(a)$. ■