Homework #2 Ch 3 (061)

$\frac{6}{67}$) G a gp, $x \in G$. If $x^2 \neq e$ and $x^6 = e$, prove $x^4 \neq e$ $x^5 \neq e$ and $|x| = ?$

suppose that $x^4 = e \Rightarrow e = x^6 = x^4 \cdot x^2 = e \cdot x^2 = x^2$

contradiction $\Rightarrow x^4 \neq e$. 1

Suppose $x^5 = e \Rightarrow e = x^6 = x^5 \cdot x = e \cdot x = x$. 1

$\Rightarrow x^2 = e$ contradiction $\Rightarrow x^5 \neq e$

$\therefore |x| = 3$ or 6. 1

$\frac{10}{67}$) let G be an Abelian gp and $a, b \in G$ st. $a^2 = e$, $b^2 = e$ $a \neq b$. $\therefore (ab)^2 = abab = a^2b^2 = e \cdot e = e$ 1

<u>Claim</u> $H = \{e, a, b, ab\} \leq G$.

$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

1 H is closed under the operation by cayley table.

$a^{-1} = a$, $b^{-1} = b$, $(ab)^{-1} = ab$ 1

$e^{-1} = e$. H is closed under inverse

So, by the two step subgp test, $H \leq G$. 1

& $\therefore |H| = 4$ we are done.

$\frac{14}{68}$) Let H and K be subgps of a gp G. $\therefore e \in H, e \in K, \Rightarrow e \in H \cap K \Rightarrow H \cap K \neq \phi$. 1

Let $a, b \in H \cap K \Rightarrow a, b \in H$ and $a, b \in K$. 1

$\Rightarrow ab^{-1} \in H$ & $ab^{-1} \in K \Rightarrow ab^{-1} \in H \cap K$

$\therefore$ By One step subgp test, $H \cap K$ is a subgp of G. 1

$\frac{20}{68}$) To prove that $C(H) = \{x \in G \mid xh = hx \ \forall h \in H\} \leq G$,

we have $C(H) \neq \emptyset$ since $e \in C(H)$ ~~also~~,

Now, let $a, b \in C(H)$ i.e. $ah = ha$ and $bh = hb \ \forall h \in H$.

$\Rightarrow b^{-1}(bh)b^{-1} = b^{-1}(hb)b^{-1} \Rightarrow hb^{-1} = b^{-1}h \ \overset{\forall h \in H}{\Rightarrow} \ b^{-1} \in C(H)$.

$\therefore (ab^{-1})h = a(b^{-1}h) = a(hb^{-1}) = (ah)b^{-1} = (ha)b^{-1} = h(ab^{-1})$

$\forall h \in H$. $\therefore ab^{-1} \in C(H)$.

$\therefore$ By the one step subgp test, $C(H) \leq G$.

$\frac{28}{69}$) $\because A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow |A| = 4$.

$\because B^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow |B| = 3$

$\because AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $(AB)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $(AB)^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, ..., $(AB)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$\therefore |AB| = \infty$.

$\frac{36}{69}$) First suppose that $H = \{a, b, e\}$ is a gp of order 3.

if $a^2 = e$, $b^2 = e$, then by Ex. 10 ch 3, H must have a subgp of order 4. Contradiction since $|H| = 3$.

$\therefore$ Either a or b is of order 3 say a.

$\therefore a^2 \neq e$, $a^3 = e$, so, $a^2 = b$ o.w. a contra. since $|H| = 3$.

$\Rightarrow b^2 = (a^2)^2 = a^4 = a \neq e \Rightarrow b^3 = (a^2)^3 = (a^3)^2 = e^2 = e$.

Now, if G is a gp that has exactly 8 elements of order 3 say $a_1, \ldots, a_8$.

Thus G has 4 subgps of order 3 since each of which will have 2 elements of order 3 and the identity.

Notice: these subgps are ~~[illegible]~~ disjoint.