# A Numerical Comparison for Coupled Boussinesq Equations by Using the ADM

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### Abstract

The solitary wave solutions of various Boussinesq systems of equations are obtained by using the decomposition method. The solutions were calculated in the form of a convergent power series with easily computable components. The convergence of the method is illustrated numerically for the system with various initial values. The present algorithm performs extremely well in terms of accuracy and simplicity.

**AMS Subject Classification:** 35Q53.

**Keywords:** The decomposition method, Coupled Boussinesq equation, Solitary wave solutions.

## 1  Introduction

In this paper, we studied solitary wave solutions for a general Boussinesq (gBQ) type fluid model both analytically and numerically. There are many types of equation form of the Boussinesq equation, one of them is given by

$$u_t + v_x + uu_x = c_1 u_{xxt}, \quad v_t + (uv)_x = c_2 u_{xxx}, \tag{1}$$

where $c_1$ and $c_2$ are real constants. This system which generalizes the classical Boussinesq equations was derived by Sachs [1] to describe small amplitude long waves in a water channel.

Exact traveling-wave solutions of the system (1) have been derived by Fan [2]. In this paper we will focus on finding analytical approximate and exact traveling wave solution of the system (1) using the Adomian decomposition method [3, 4]. The method provides the solutions in the form of a series with easily computable terms. The accuracy and rapid convergence of the solutions are demonstrated through some numerical examples.

## 2    Outline of the decomposition method

To solve the nonlinear Boussinesq equations (1) we write the system of the partial differential equations (1) in an operator form

$$\mathcal{L}_t u + v_x + M(u, u_x) = c_1 \mathcal{L}_x(u_t), \quad \mathcal{L}_t v + (N(u, v))_x = c_2 \mathcal{L}_x(u_x), \tag{2}$$

where the notation $\mathcal{L}_t = \frac{\partial}{\partial t}$ and $\mathcal{L}_x = \frac{\partial^2}{\partial x^2}$ symbolizes the linear differential operators, the notations $M(u, u_x) = u u_x$ and $N(u, v) = uv$ symbolize the nonlinear operators. Applying the inverse operator $\mathcal{L}_t^{-1} = \int_0^t (\cdot)\, dt$ to the system (2) yields

$$
\begin{aligned}
u(x, t) &= g_1(x) - \mathcal{L}_t^{-1} \left[ v_x + M(u, u_x) - c_1 \mathcal{L}_x(u_t) \right], \\
v(x, t) &= g_2(x) - \mathcal{L}_t^{-1} \left[ (N(u, v))_x - c_2 \mathcal{L}_x(u_x) \right],
\end{aligned}
\tag{3}
$$

where $g_1(x) = u(x, 0)$ and $g_2(x) = v(x, 0)$ are given functions for initial conditions. The Adomian decomposition method [3]–[5] assumes an infinite series solution for unknown functions $u(x, t)$ and $v(x, t)$ in the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \tag{4}$$

and the nonlinear operators $M(u, u_x) = u u_x$ and $N(u, v) = uv$ by the infinite series of Adomian polynomials given by

$$M(u, u_x) = \sum_{n=0}^{\infty} A_n, \quad N(u, v) = \sum_{n=0}^{\infty} B_n, \tag{5}$$

where $A_n$ and $B_n$ are the appropriate Adomian's polynomials which are generated according to algorithms determined in Refs. [3]–[6]. For the nonlinear operator $M(u, u_x)$ these polynomials can be defined by

$$A_n(u_0, \ldots, u_n; (u_0)_x, \ldots, (u_n)_x) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} M \left( \sum_{k=0}^{n} \lambda^k u_k, \sum_{k=0}^{n} \lambda^k (u_k)_x \right) \right]_{\lambda=0}, \tag{6}$$

$$B_n(u_0, \ldots, u_n; v_0, \ldots, v_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{k=0}^{n} \lambda^k u_k, \sum_{k=0}^{n} \lambda^k v_k \right) \right]_{\lambda=0}, \tag{7}$$

$$n \geq 0.$$

This formula is easy to be set in a computer code to get as many polynomial as we need in the calculation of the numerical as well as explicit solutions. For a detailed explanation of Adomian decomposition method and other general formula of Adomian polynomials, we refer the reader to [3, 6].

Following the modified decomposition method [5], the series (4) are constructed by the following recursive relations

$$u_0(x,t) = 0.0, \quad u_1(x,t) = g_1(x) - \mathcal{L}_t^{-1}\left[(v_0)_x + A_0 - c_1\mathcal{L}_x((u_0)_t)\right],$$

$$u_{n+1}(x,t) = -\mathcal{L}_t^{-1}\left[(v_n)_x + A_n - c_1\mathcal{L}_x((u_n)_t)\right],$$

$$v_0(x,t) = 0.0, \quad v_0(x,t) = g_2(x) - \mathcal{L}_t^{-1}\left[(B_0)_x - c_2\mathcal{L}_x((u_0)_x)\right],$$

$$v_{n+1}(x,t) = -\mathcal{L}_t^{-1}\left[(B_n)_x - c_2\mathcal{L}_x((u_n)_x)\right], \tag{8}$$

where $n \geq 1$, the functions $g_1(x)$ and $g_2(x)$ are taken from the initial conditions. It is worth noting that once the zero components $u_0$ and $v_0$ are defined, then the remaining components $u_n$ and $v_n$, $n \geq 1$, can be completely determined so that each pair of terms are computed by using the previous terms. As a result, the components $u_0$, $u_1$, $u_2$, ... and $v_0$, $v_1$, $v_2$, ... are identified and the series solutions thus entirely determined. However, in many cases the exact solution in a closed form may be obtained.

For numerical comparisons purposes, we construct the solution $u(x,t)$ and $v(x,t)$

$$\lim_{n\to\infty} \phi_n = u(x,t), \quad \lim_{n\to\infty} \varphi_n = v(x,t), \tag{9}$$

where $\phi_n(x,t) = \sum_{k=0}^{n} u_k(x,t)$, $\varphi_n(x,t) = \sum_{k=0}^{n} v_k(x,t)$, $n \geq 0$, and the recurrence relation is given as in (8). Moreover, the convergence of the decomposition series has been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [7]–[13]. They obtained some results about the speed of convergence of this method. In a recent work of Ngarhasta *et al.* [14] they have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [14]. In this work, we demonstrate how approximate solutions of the gBQ equation system are close to the corresponding exact solutions. They obtained some results about the speed of convergence of this method. To give a clear overview of the methodology, some examples will be discussed in the following section.

## 3  Implementation of the method

In this section we will be concerned with the traveling wave solutions of the gBQ equation (1) with the initial conditions

$$u(x,0) = a_0 - K_1 \tanh^2(Rx), \quad v(x,0) = b_0 - K_2 \tanh^2(Rx), \tag{10}$$

where

$$a_0 = \frac{a_2^2 + 8c_1ba_2^2 + 72c_1c_2}{12c_1a_2}, \quad b_0 = \frac{4(9c_2^2 - bc_1a_2^2)}{a_2^2}, \quad K_1 = a_2b, \quad K_2 = 6c_2b, \quad R = \sqrt{-b},$$

$b$ and $a_2$ being arbitrary constants.

Using (8) with (6)–(7) for the functional coupled equation (1) and initial conditions (10) gives

$$
\begin{aligned}
u_0 &= 0.0, \; v_0 = 0.0, \; u_1 = a_0 - K_1 \tanh^2(R\,x), \; v_1 = b_0 - K_2 \tanh^2(R\,x), \quad (11) \\
u_2 &= -2\,K_2\,R\,t\,\mathrm{sech}^2(R\,x)\,\tanh(R\,x), \quad (12) \\
v_2 &= c_2 t \left(16 K_1 R^3 \mathrm{sech}(Rx)^4 \tanh(Rx) - 8 K_1 R^3 \mathrm{sech}^2(Rx) \tanh^3(Rx)\right), \quad (13) \\
u_3 &= c_2\,K_1\,R^4\,t^2 \left(33 - 26\cosh(2\,R\,x) + \cosh(4\,R\,x)\right)\mathrm{sech}^6(R\,x) \\
    &\quad + R\,t\left[-a_0\,K_1 - K_1^2 + 20\,c_1\,K_2\,R^2 - a_0\,K_1\,\cosh(2\,R\,x) + K_1^2\,\cosh(2\,R\,x)\right. \\
    &\quad \left. - 4\,c_1\,K_2\,R^2\,\cosh(2\,R\,x)\right]\mathrm{sech}^4(R\,x)\,\tanh(R\,x), \quad (14) \\
v_3 &= c_2\,K_2\,R^4\,t^2 \left(33 - 26\cosh(2\,R\,x) + \cosh(4\,R\,x)\right)\mathrm{sech}^6(R\,x) \\
    &\quad - R\,t\left[b_0\,K_1 + a_0\,K_2 + 2\,K_1\,K_2 + b_0\,K_1\,\cosh(2\,R\,x) + a_0\,K_2\,\cosh(2\,R\,x)\right. \\
    &\quad \left. - 2\,K_1\,K_2\,\cosh(2\,R\,x)\right]\mathrm{sech}^4(R\,x)\,\tanh(R\,x), \quad (15)
\end{aligned}
$$

and so on, the other components of the decomposition series (4) can be determined in a similar way. Substituting (11)–(15) into (4) and using the decomposition series (4) which is a Taylor series, we obtain the closed form solutions

$$
u(x,t) = a_0 - K_1 \tanh^2\left(R\,(x - c\,t)\right), \quad v(x,t) = b_0 - K_2 \tanh^2\left(R\,(x - c\,t)\right), \quad (16)
$$

where

$$
a_0 = \frac{a_2^2 + 8c_1 b a_2^2 + 72 c_1 c_2}{12 c_1 a_2}, \; b_0 = \frac{4(9c_2^2 - bc_1 a_2^2)}{a_2^2}, \; K_1 = a_2 b, \; K_2 = 6 c_2 b, \; c = \frac{a_2}{12 c_1},
$$

$R = \sqrt{-b}$, $b$ and $a_2$ being arbitrary constants. These solutions are constructed by Fan [2].

# 4    Numerical results and discussions

For numerical comparisons purposes, we construct the general form of the solution $u(x,t)$ and $v(x,t)$ by using formulae (9). In order to see the accuracy of the solutions by using the decomposition method, we consider various values of the wave speed $c$.

Furthermore, as the decomposition method does not require discretization of the variables, *i.e.*, time and space, it is not effected by computation roundoff errors and one is not faced with the necessity of large computer memory and time. The accuracy of the decomposition method for the gBQ equation is controllable and the absolute errors are very small which are given in Table 1. It shows that the

implemented method achieves an accuracy of minimum six and maximum seven significant figures for the Eq. (1) for initial conditions (10) using a reasonable small value of $n$ in the formulae (8) with (6)–(7). The absolute values of the differences $b$ between the numerical exact solutions and the approximate solutions obtained as $n = 4$ by using the formulae (10) are given in Table 1 for the Eq. (1) for the initial conditions (10). There is no visible difference in the two solutions if the values of $b$ are small. This is the nature of the series method. It is also evident that when more terms for the decomposition series are computed, the numerical results are getting much closer to the corresponding exact solutions with the initial conditions (10) of the Eq. (1).

| $t_i \backslash x_i$ | $\|u - \phi_n\|$ | | | | |
|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| 0.1 | 8.51262E-08 | 3.16170E-07 | 6.93114E-07 | 1.21594E-06 | 1.88464E-06 |
| 0.2 | 9.72753E-08 | 3.40433E-07 | 7.29944E-07 | 1.26426E-06 | 1.94486E-06 |
| 0.3 | 1.09417E-07 | 3.64670E-07 | 7.65707E-07 | 1.31248E-06 | 2.00493E-06 |
| 0.4 | 1.21549E-07 | 3.88876E-07 | 8.01913E-07 | 1.36059E-06 | 2.06484E-06 |
| 0.5 | 1.33672E-07 | 4.13052E-07 | 8.38055E-07 | 1.40859E-06 | 2.12458E-06 |
| | $\|v - \varphi_n\|$ | | | | |
| 0.1 | 5.10934E-07 | 1.89774E-06 | 4.16041E-06 | 7.29896E-06 | 1.13134E-05 |
| 0.2 | 5.83874E-07 | 2.04351E-06 | 4.37892E-06 | 7.59009E-06 | 1.16770E-05 |
| 0.3 | 6.56767E-07 | 2.18912E-06 | 4.59707E-06 | 7.88061E-06 | 1.20397E-05 |
| 0.4 | 7.29609E-07 | 2.33456E-06 | 4.81486E-06 | 8.17050E-06 | 1.24015E-05 |
| 0.5 | 8.02391E-07 | 2.47981E-06 | 5.03226E-06 | 8.45974E-06 | 1.27632E-05 |

*Table 1. The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the analytical solution (16) for $u(x,t)$ and $v(x,t)$ when $c_1 = 0.05$, $c_2 = 0.05$, $b_1 = -0.001$ and $a_2 = 0.05$ for the traveling wave solution of the Eqn. (1).*

Numerical approximations show a high degree of accuracy and in most cases $\phi_n$ and $\varphi_n$, the $n$-term approximations for $u$ and $v$, respectively, are accurate for quite low values of $n$. The numerical results obtained justify the advantage of this methodology. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

Furthermore, as the decomposition method does not require discretization of the variables, *i.e.*, time and space, it is not effected by computation roundoff errors and one is not faced with necessity of large computer memory and time. The accuracy of the decomposition method for the coupled nonlinear equations controllable and absolute errors is very small with the present choice of $t$ and $x$.

# 5   Conclusions

In conclusion, the Adomian decomposition method was used for finding the exact and approximate traveling-waves solutions of the gBQ equation. The method can be also easily extended to other nonlinear evaluation equations, with the aid of Mathematica (or Matlab, Maple, Reduce, etc.), the course of solving nonlinear evaluation equations can be carried out in computer. One coupled nonlinear equation with initial conditions is discussed as a demonstration. It may be concluded that the Adomian methodology is very powerful and efficient technique in finding exact solutions for wide classes of problems. It is also worth noting to point out that the advantage of the decomposition methodology is the fast convergence of the solutions.

Clearly, the series solution methodology can also be applied to many other nonlinear problems. However, as we have seen in the previous sections, the decomposition method does not require linearization or perturbation for obtaining closed form solutions. Additionally, it does not need any discretization to get numerical solutions. Clearly, the series solution methodology can be applied to various types of linear or nonlinear ordinary and partial differential equations [15]–[21].

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