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Boundary Value Problem for Functional Differential Equations

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Abstract

Suitable periodic boundary conditions for the functional differential equation $\dot{x}(t) = f(t, x, x_t)$ are conditions of the form $x_0(\theta) = x_{2\pi}(\theta)$. In this talk we use the notion of upper and lower solutions combined with monotone iterative method to prove the existence of solutions of this periodic boundary value problem.

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1 Introduction

Let us consider the boundary value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t), x_t), \\ x_0(\theta) = x_{2\pi}(\theta), \qquad \theta \in [-r, 0], \end{cases} \tag{1}$$

where $f : I \times \mathbb{R} \times C \to \mathbb{R}$ is a continuous function, $I = [0, 2\pi]$ and $C = C([-r, 0], \mathbb{R})$ is the space of continuous real valued functions defined on $[-r, 0]$. The set C is a Banach space with supremum norm $\|\phi\| = \sup_{t\in[-r,0]} |\phi(t)|$.

To our knowledge none of the methods used previously (see for example [1, 3] and the references therein) works for such boundary value problems. The aim of this talk is to give an approach based on iterative methods [2] combined with the method of upper and lower solutions.

Definition 1.1 $\alpha(t)$ is called a *lower solution* of (1) if

$$\begin{cases} \dot{\alpha}(t) \le f(t, \alpha(t), \alpha_t), & t \in [0, 2\pi], \\ \alpha(\theta) = \alpha_{2\pi}(\theta), & \forall\theta \in [-r, 0]. \end{cases}$$

$\beta(t)$ is called an *upper solution* of (1) if

$$\begin{cases} \dot{\beta}(t) \geq f(t, \beta(t), \beta_t), & t \in [0, 2\pi], \\ \beta(\theta) = \beta_{2\pi}(\theta), & \forall \theta \in [-r, 0]. \end{cases}$$

2 Main Result

Let us make the following assumption on f:

(H) $f(t, u, \phi) + Mu$ is monotone nondecreasing.

Let α and β be respectively a lower and an upper solution of (1) such that the assumption (H) is fulfilled. Define the sequence $(u_n)_{n\geq 0}$ on $C([-r, 2\pi], \mathbb{R})$ by

$$\begin{cases} u_0 = \alpha, \\ \dot{u}_{n+1} + Mu_{n+1} = f(t, u_n, u_{n,t}) + Mu_n, \\ u_{n+1,0}(\theta) = u_{n,2\pi}(\theta), \quad \theta \in [-r, 0]. \end{cases}$$

Since f is continuous and by setting $g_n(t) = f(t, u_n, u_{n,t}) + Mu_n$, we can see by recurrence that the sequence (u_n) is well defined.

Remark 2.1 We point out that the sequence (u_n) is different from the sequences used in [1, 3] and [4], since the first boundary condition (or the initial condition) $u_{n+1,0}$ is expressed in term of the previous term of the sequence at the second boundary condition $u_{n,2\pi}$, *i.e.*, $u_{n+1,0}(\theta) = u_{n,2\pi}(\theta) \quad \theta \in [-r, 0]$.

If α and β are a lower and an upper solution of (1) such that the assumption (H) is fullfilled, then the sequence (u_n) has the following property:

Proposition 2.2 *For all $k \in \mathbb{N}$ one has*

$$\alpha(t) \leq u_k(t) \leq \beta(t).$$

Proof. We prove that $u_k(t) \leq \beta(t) \quad \forall k \in \mathbb{N}$, the proof for the left inequality is analogous. We proceed by recurrence.

For $k = 0$ one has $u_0(t) = \alpha(t) \leq \beta(t) \quad \forall t \in [-r, 2\pi]$.

Now we suppose that $u_k(t) \leq \beta(t)$ and show that $u_{k+1}(t) \leq \beta(t) \quad \forall t \in [-r, 2\pi]$. Since

$$\dot{u}_{k+1} + Mu_{k+1} = f(t, u_k, u_{k,t}) + Mu_k$$

and

$$\dot{\beta} + M\beta \geq f(t, \beta, \beta_t) + M\beta,$$

one has

$$\begin{aligned}(u_{k+1}-\beta)' + M(u_{k+1}-\beta) &\leq [f(t,u_k,u_{k,t}) + Mu_k] - [f(t,\beta,\beta_t) + M\beta] \\ &= f(t,u_k,u_{k,t}) - f(t,\beta,\beta_t) + M(u_k-\beta).\end{aligned}$$

By application of (H) we obtain

$$(u_{k+1}-\beta)' + M(u_{k+1}-\beta) \leq 0.$$

Moreover,

$$u_{k+1}(0) - \beta(0) = u_k(2\pi) - \beta(2\pi) \leq 0.$$

Then we have a problem of the form

$$\begin{cases} \dot{w} + Mw \leq 0 & \forall t \in]0,2\pi[, \\ w(0) \leq 0, \end{cases}$$

where $w = u_{k+1} - \beta$. Hence the question is to prove that $w(t) \leq 0$ whenever $t \in [0,2\pi]$. To this end one has

$$\begin{aligned}(e^{Mt}w(t))' &= Me^{Mt}w(t) + e^{Mt}\dot{w}(t) \\ &= e^{Mt}[\dot{w}(t) + Mw(t)] \\ &\leq 0.\end{aligned}$$

By integrating the term $(e^{Mt}w(t))' \leq 0$ we get

$$\begin{aligned} e^{Mt}w(t) - e^0 w(0) &\leq 0, \\ e^{Mt}w(t) &\leq w(0), \\ e^{Mt}w(t) &\leq 0, \end{aligned}$$

which is equivalent to $w \leq 0$. This ends the proof since $w = u_{k+1} - \beta$. ■

Theorem 2.3 *The sequence* $(u_k)_{k\in\mathbb{N}}$ *has a convergent subsequence, which converges to a solution* u *of the problem* (1), *i.e.,* $u \in C([-r,2\pi],\mathbb{R}) \cap C^1([0,2\pi],\mathbb{R})$ *and* $u_0(\theta) = u_{2\pi}(\theta)$.

Proof. Since $\alpha \leq u_k \leq \beta$, one has

$$\|u_k\| \leq \|\alpha\| + \|\beta\| \leq c.$$

In addition,

$$\begin{aligned} \dot{u}_k + Mu_k &= f(t,u_{k-1},u_{k-1,t}) + Mu_{k-1} \text{ and} \\ \|\dot{u}_k\| &\leq M\|u_k\| + \sup_{\substack{\alpha\leq u\leq\beta \\ t\in[0,2\pi]}} \|f(t,u_{k-1},u_{k-1,t})\| + M\|u_{k-1}\|, \\ \|\dot{u}_k\| &\leq K. \end{aligned}$$

Hence the sequence (u_k) is equicontinuous and since it is bounded, by the Ascoli-Arzela theorem the sequence (u_k) has a convergent subsequence.

From

$$\dot{u}_{n+1} + Mu_{n+1} = f(t, u_n, u_{n,t}) + Mu_n,$$
$$u_{n+1,0}(\theta) = u_{n,2\pi}(\theta), \quad \theta \in [-r, 0],$$

an integration yields

$$u_{n+1}(t) - u_{n+1}(0) = \int_0^t f(s, u_n, u_{n,s})\, ds + M \int_0^t (u_n(s) - u_{n+1}(s))\, ds.$$

Since $|f(t, u_{k-1}, u_{k-1,t})| \leq \sup\limits_{\substack{\alpha \leq v \leq \beta \\ t \in [0,2\pi]}} \{|f(t, v, v_t)|\} \leq L,$ L is a constant, by the dominated convergence theorem of Lebesgue we get

$$u(t) - u(0) = \int_0^t f(s, u, u_s)\, ds.$$

In addition, one has

$$u_0(\theta) = u_{2\pi}(\theta) \quad \forall \theta[-r, 0].$$

This ends the proof. ■

3 Application

As an application we consider the following functional differential equation

$$\dot{x}(t) = g(t, x_t) - Kx(t)$$

with g continuous and K is a positive constant. Let $m(\alpha) = \inf\limits_{0 \leq t \leq 2\pi} g(t, \bar{\alpha})$ and $M(\alpha) = \sup\limits_{0 \leq t \leq 2\pi} g(t, \bar{\alpha})$, where $\bar{\alpha}$ is the constant function equal to α. Let us make the hypotheses $m_0 = \limsup\limits_{\alpha \to -\infty} \frac{|m(\alpha)|}{|\alpha|} < +\infty$ and $M_0 = \limsup\limits_{\alpha \to +\infty} \frac{M(\alpha)}{\alpha} < +\infty$.

Proposition 3.1 *If the function $g(t, \cdot)$ is monotone nondecreasing, for all $K >$ $\max(m_0, M_0)$ the boundary value problem*

$$\begin{cases} \dot{x}(t) = g(t, x_t) - Kx(t), \\ x_0(\theta) = x_{2\pi}(\theta), \qquad \theta \in [-r, 0], \end{cases} \tag{2}$$

has at least one solution.

Proof. The equation (2) is of the form of the equation (1) with

$$f(t,u,\varphi) := g(t,\varphi) - ku.$$

Since g is monotone nondecreasing, it is easy to verify that for $M \geq K$ the function $g(t,\varphi) + (M - K)u$ satisfies the Hypothesis (H). Let α_0 be a large negative real number. The constant solution equal to α_0 is a lower solution of the equation (2). Indeed, it is enough to check that

$$g(t,\alpha_0) - K\alpha_0 \geq 0.$$

This follows from $K \geq m_0 = \limsup_{\alpha\to-\infty} \frac{|m(\alpha)|}{|\alpha|} \geq \frac{|m(\alpha_0)|}{|\alpha_0|}$.

We prove in the same way that the constant solution β equal to β_0 — a positive real number — is an upper solution. ■

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