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The Center-Focus Problem and Variational Equations

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1 Introduction

The center-focus (CF) problem for plane ODE systems with polynomial right-hand sides presents so formidable a challenge that its complete solution is not expected in the foreseeable future. In fact, until recently, this problem was solved only for a few types of systems with one edge of the Newton polygon [1, 2]. These subtypes are systems with a nondegenerate linear part; systems with a Jordan cell linear part; and systems with a homogeneous nondegenerate truncation.

The existing methods of solution of CF problem for these systems may be described as either a reduction to normal forms (including methods using formal power series [6]) or the construction of the formal first integral [5]. The focal values obtained by these methods usually lose some information on the asymptotics of the Poincaré mapping and provide only the center conditions.

In this paper, we present an approach to the CF problem based on the generalization of the notion of variational equations, where the necessary conditions for the center are obtained as a byproduct along with the asymptotics of the Poincaré mapping.

This approach allows to obtain a simple algorithm for computation of the exact focal values for all types of systems with one edge of the Newton polygon. We note for comparison that a brief description of an algorithm based on the proof by Poincaré of the existence of the first analytical integral takes a few pages in [5]; and it is applicable only for systems with nondegenerate linear part. Our algorithm described below takes just a few lines.

The benefit of computing explicit asymptotics of the Poincaré mapping becomes evident when we consider a related problem of degenerate cycle generation. We give an example illustrating this application.

Finally, we give a simple proof of almost algebraic [4] solvability of CF problem for all types of systems with one edge of the Newton polygon.

The structure of this paper is as follows. In Section 2 we describe the algorithm based on variational equations of high order which resolves CF problem for monodromic singularities with analytical Poincaré mapping. We give a simple proof of the algorithm, which is easily adaptable for computer algebra systems.

In Section 3 we introduce generalized polar changes of variables and demonstrate that they are applicable to all types of systems with one edge of the Newton polygon. Thus the algorithm in Section 2 can be applied to all cases ever studied by various methods. We define a quasi-homogeneous degree of a monomial in polynomial systems such that coefficients of monomials of the same degree take part in variational equations of the same order.

In Section 4 we consider some applications of the theory and suggest a generalization which applies to systems with two edges of the Newton polygon. Such systems were never studied in the framework of CF problem.

2 Variational equations

In this paper, we study systems of equations

$$dx/dt \stackrel{\text{def}}{=} \dot{x} = p(x,y), \quad dy/dt \stackrel{\text{def}}{=} \dot{y} = q(x,y), \tag{2.1}$$

where p and q are polynomials and the origin $O = (0,0)$ is a monodromic stationary point of the system (2.1). Suppose that there exists a change of variables $x = x(r,\varphi)$, $y = y(r,\varphi)$ such that the integral curves of the system (2.1) coincide with the integral curves of the equation

$$dr/d\varphi \stackrel{\text{def}}{=} r' = f(r,\varphi) = f(r,\varphi + 2\pi). \tag{2.2}$$

We assume that the Poincaré mapping $r(0) \to r(2\pi)$ is analytical at the origin.

The algorithm for computing variational equations of arbitrary order for the equation (2.2) is as follows [8, 9]. We write this equation as a variational equation of zero order substituting $r = r_0$, *i.e.*, $r_0' = f(r_0,\varphi)$. Let $r_0(\varphi)$ be the solution to this equation with the initial value $\rho = r_0(0) \neq 0$. We introduce the notation $r_k(\varphi) = \partial r_{k-1}(\varphi)/\partial\rho$, $k = 1,2,\ldots$ By formal differentiation we find the variational equations $r_k' = \partial f_{k-1}(r_0,\ldots,r_{k-1},\varphi)/\partial\rho = f_k(r_0,\ldots,r_k,\varphi) = \partial^k f(r_0,\varphi)/\partial\rho^k$. The initial values for the solutions $r_k(\varphi)$ to these equations are $r_1(0) = 1$, and $r_k(0) = 0$, $k = 2,3,\ldots$

The procedure to resolve CF problem for the equation (2.2) is to find solutions to all variational equations on the trivial solution $r_0 \equiv 0$. Then $r_k(2\pi)$, $k = 1,2,\ldots$, are the focal values. This follows from the following theorem.

Theorem 2.1 *Let the Poincaré mapping $r(0) \to r(2\pi)$ for the equation (2.2) be analytical at the origin. The origin is a center if and only if all solutions to the variational equations on the trivial solution $r_0 \equiv 0$ are 2π-periodic bounded functions.*

Proof. Consider the *characteristic set* of the equation (2.2) $\chi = \{r(0), r(2\pi)\}$ [8, 9] for sufficiently small initial values of $r(0)$. The set χ is an analytical curve passing through the origin. The origin is the center if and only if the set χ is the bisectrix of the first quadrant of the plane $\mathbb{R}^2$.

Let $r_k(\varphi)$, $k = 1, 2, \ldots$, be the solutions to the variational equations and $\rho = r_0(0) \neq 0$ be the initial value of the solution to the equation (2.2). The values $r_k(2\pi)$, $k = 1, 2, \ldots$, are the derivatives to the curve χ at the point ρ [8] (see Fig. 1). All solutions to the variational equations depend analytically on the initial value ρ by the data. Hence the values $r_k(2\pi)$, $k = 1, 2, \ldots$, for the variational equations on the trivial solution $r_0 \equiv 0$ are the derivatives to the curve χ at the origin.

An analytical curve is the bisectrix of the first quadrant of the plane $\mathbb{R}^2$ if and only if it has the same derivatives at the origin, namely, $r_1(2\pi) = 1$, $r_k(2\pi) = 0$, $k = 2, 3, \ldots$ These values coincide with the initial values of the solutions to the variational equations; hence the sufficient condition is proved.

Further, we will need only variational equations on the trivial solution.

Now we prove the necessity. The first variational equation has the form

$$r_1' = r_1 f_1, \tag{2.3}$$

where $f_1(\varphi)$ is a bounded 2π-periodic function. Hence if $r_1(2\pi) = 1$, then $r_1(\varphi)$ is also a 2π-periodic bounded function. In fact,

$$r_1(\varphi) = \exp\left(\int_0^{\varphi} f_1(\phi)\, d\phi\right). \tag{2.4}$$

By induction, the n-th variational equation has the form

$$r_n' = r_n f_1 + r_1^2 g_n, \quad n = 2, 3, \ldots, \tag{2.5}$$

where $g_n(\varphi)$ is a bounded 2π-periodic function expressed through previously determined solutions to variational equations. Hence if $r_n(2\pi) = 0$, then $r_n(\varphi)$ is also a 2π-periodic bounded function. In fact,

$$r_n(\varphi) = r_1(\varphi) \int_0^{\varphi} r_1(\phi) g_k(\phi)\, d\phi. \tag{2.6}$$

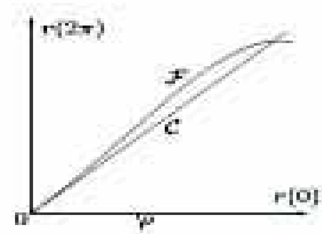

Fig. 1. Characteristic sets of the center ($\mathcal{C}$) and the focus ($\mathcal{F}$).

Theorem 2.1 is proved along with the fact that all variational equations are explicitly integrable. ■

Thus, Theorem 2.1 provides the explicit expansion of the Poincaré mapping for the equation (2.2) at the origin in Taylor series:

$$r(2\pi) = r_1(2\pi)r(0) + \frac{r_2(2\pi)r(0)^2}{2!} + \cdots \tag{2.7}$$

Before we move further, let us examine what this algorithm gives for systems (2.1) with nondegenerate linear part, which, historically, is the most studied case of CF problem [3].

Let the linear part of the system (2.1) have imaginary eigenvalues. Otherwise the system has a structurally stable focus and the algorithm above takes one step. By the ordinary polar change of variables we reduce the system to the form (2.2). Then we use formal differentiation and obtain some variational equations. Then we substitute $r_0 = 0$ in them. We have then $r_1(\varphi) \equiv 1$, and all subsequent solutions to variational equations are trigonometric polynomials. All secular terms may be omitted by Theorem 2.1, since they automatically vanish if the previously determined center conditions are satisfied.

Unlike the computation of the formal first integral [5], there are no arbitrary values involved in the focal ones. So they may be used for analysis of the cycle generation (see Section 4).

3 Generalized polar changes

We consider ODE systems (2.1) that have one edge of the Newton polygon and that by renormalization of the variables x, y, and t may be written as

$$\begin{aligned} \dot{x} &= p(x,y) = \quad y^{2jm-1} + \cdots \,, \\ \dot{y} &= q(x,y) = -x^{2jn-1} + \cdots \,, \end{aligned} \tag{3.1}$$

where $j, m, n \in \mathbb{N}$; m and n are relative primes; and all monomials denoted by dots map on the edge and/or inside the Newton polygon of the system (3.1).

Systems (3.1) include all types that were ever studied in the framework of CF problem [1, 2], namely, systems with nondegenerate linear part $j = m = n = 1$; systems with a Jordan cell linear part $j = m = 1$; and systems with a homogeneous nondegenerate truncation $m = n$.

We suggest the following changes of variables

$$\begin{aligned} x &= \alpha\, r^m\, |\cos\varphi|^k \operatorname{sign}(\cos\varphi), \\ y &= \beta\, r^n\, |\sin\varphi|^\ell \operatorname{sign}(\sin\varphi), \end{aligned} \tag{3.2}$$

where m and n are the same as in the system (3.1), and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $k \in \mathbb{Q}$, $\ell \in \mathbb{Q}$ are positive constants. The system (3.1) takes the form

$$r' = \frac{r\,(x\,k\,q(x,y)\sin^2\varphi + y\,\ell\,p(x,y)\cos^2\varphi)}{\sin\varphi\,\cos\varphi\,(x\,m\,q(x,y) - y\,n\,p(x,y))}. \tag{3.3}$$

The changes of variables (3.2) are applied in four curvilinear sectors of the plane $(i-1)\pi/2 \le \varphi \le i\pi/2$, $i = 1, \ldots, 4$, and the equation (3.3) corresponds to four ODEs defined in each sector. The Poincaré mapping is matched from the four different representations of transient trajectories in these sectors.

The changes (3.2) are more general than we need for Theorem 2.1. They will be used later. The constants α, β, k, ℓ may be chosen in such a way that the trajectory of the *truncated system*, *i.e.*, the system having monomials only on the edge of the Newton polygon [2], simplifies.

But first we prove that the changes (3.2) give an analytical Poincaré mapping for systems (3.1) if certain conditions on the coefficients of the truncated system are satisfied. For that in this section we take $\alpha = \beta = k = \ell = 1$ in (3.2).

Let the *quasi-homogeneous degree* of the monomial $x^k y^\ell$ $(k, \ell \in \mathbb{N})$ be $km + (\ell+1)n$ in the first equation of the system (3.1), and be $(k+1)m + \ell n$ in the second equation of the system (3.1).

It is obvious that the truncated system (3.1) is quasi-homogeneous and that all monomials mapping on the edge of the Newton polygon are, in a sense, of the same order of smallness. Here their degree is $2jmn$. But, surprisingly, the order of smallness of all monomials in systems (3.1) was never defined before. The definition above (yet to be proved correct) is not unique, but it sorts out all monomials in the system (not only for monodromic stationary points, by the way).

We define the function

$$F(\varphi) = \lim_{r \to 0} \frac{r'}{r}, \tag{3.4}$$

where r' is the right-hand side of the equation (3.3).

Theorem 3.1 *If the denominator of the function $F(\varphi)$ does not vanish for sufficiently small $|\mathrm{Im}\,\varphi|$, then the stationary point O of the system* (3.1) *is monodromic, and the Poincaré mapping is analytical at the origin.*

Proof. The numerator and the denominator of the function $F(\varphi)$ are trigonometric polynomials with coefficients of monomials of degree $2jmn$ in the system (3.1). This corresponds to the fact that the truncated system being quasi-homogeneous is integrable by the substitution (3.2).

The denominator of the function $F(\varphi)$ satisfies the condition of the theorem for sufficiently small coefficients of the truncated system, since if they vanish, then the truncated system (3.1) is a Hamiltonian one.

We expand the right-hand side of the equation (3.3) in Taylor series in powers of r. Each term of the expansion is a fraction with the denominator being some power of the denominator of the function $F(\varphi)$. Hence the right-hand side of the equation (3.3) is analytical for sufficiently small $|r|$ and $|\mathrm{Im}\,\varphi|$, and hence the solution to ODE (3.3) exists for $\varphi \in [0, 2\pi]$, *i.e.*, the stationary point O of the system (3.1) is monodromic. ∎

For example, the system

$$\dot{x} = y + ax^2, \quad \dot{y} = -x^3$$

with $j = 1$, $m = 1$, $n = 2$ has

$$F(\varphi) = -\frac{\cos\varphi(a\cos^2\varphi + \sin^3\varphi)}{\cos^4\varphi + 2a\cos^2\varphi\sin\varphi + 2\sin^2\varphi}.$$

The denominator $\cos^4\varphi + 2a\cos^2\varphi\sin\varphi + 2\sin^2\varphi \neq 0$ for $|a| < \sqrt{2}$.

Theorem 3.2 *The variational equation of the order N computed for the equation* (3.3) *depends only on coefficients of monomials in the system* (3.1) *of degree no greater than $2jmn + N - 1$.*

Proof Consider a monomial $x^k y^\ell$ in the first equation (3.1). The power of r in the equation (3.3) corresponding to this monomial equals the degree of the monomial plus 1 in the numerator and equals the degree of the monomial in the denominator of the equation (3.3). The same is true for monomials in the second equation (3.1). So coefficients of all monomials of the same degree are gathered at the same powers of r in the equation (3.3). We divide the numerator and denominator of the equation (3.3) by r^{2jmn} and multiply both sides of the equation (3.3) by the denominator. Then we use the procedure for computation of variational equations. Each formal differentiation decreases the powers of r_0 by 1. So when we substitute $r_0' = r_0 = 0$, the first variational equation would depend only on coefficients of monomials of the truncated system (3.1), *i.e.*, monomials of degree $2jmn$, etc. ■

4 Some applications

It follows from Theorem 2.1 that the values $r_k(2\pi)$, $k = 1, 2, \ldots$, are the focal ones for the monodromic stationary point O of the system (3.1). If $r_1(2\pi) \neq 1$, then the point O is a focus of the *first order*. It is an analog of the structurally stable focus, since it persists if there are small changes of the coefficients of the monomials of the truncated system, *i.e.*, the curve χ is transversal to the bisectrix $\mathcal{C}$ at the origin. If $r_1(2\pi) = 1$, $r_\ell(2\pi) = 0$, $\ell < k$, and $r_k(2\pi) \neq 0$, then it is a focus of the *order* k. It is an analog of the weak focus. Obviously, the order of the focus coincides with the order of tangency of the curves χ and $\mathcal{C}$ at the origin.

Theorem 3.2 allows to estimate the order of the focus for the monodromic stationary point without computing the variational equations. Indeed, omitting monomials of the degree greater than $2jmn + N - 1$ cannot affect the focal values of the order less or equal to $N = 1, 2, \ldots$

We remark that the disposition of the curve χ with respect to the bisectrix $\mathcal{C}$ allows to make some conclusions on the nature of solutions to the system (3.1) in a small neighborhood of the stationary point O. For example, Fig. 1 corresponds to an unstable weak focus and stable limit cycle in the equation (3.3). In systems (3.1) stability and instability interchange, since the trajectories go clockwise there.

When we consider a system (3.1) depending on a parameter, then the parameter change may cause the characteristic curve χ to twist and intersect with the bisectrix $\mathcal{C}$. This corresponds to the limit cycle generation. The Hopf bifurcation is the simplest example of this phenomenon; here the first focal value becomes equal to 1. When focal values of higher order vanish, there are possible bifurcations of arbitrary high degeneracy. The degeneracy may be measured by the number of variational equations needed to resolve it.

Example 4.1. Consider the following system of equations

$$\dot{x} = y^3 + a\,x^3 y + b\,x^5, \quad \dot{y} = -x^5 + c\,x^2 y^2. \tag{4.1}$$

Such systems were never considered in the framework of CF problem. First, we make the change of variables (3.2) with $m = 2$, $n = 3$, and $\alpha = \beta = k = \ell = 1$. The function $F(\varphi)$ (3.4) for the system (4.1) takes the form

$$F(\varphi) = -\frac{\sin\varphi\cos\varphi(\sin^2\varphi + a\,\cos^3\varphi + c\,\sin^2\varphi\cos\varphi - \cos^4\varphi)}{2\cos^6\varphi + (3a - 2c)\sin^2\varphi\cos^3\varphi + 3\sin^4\varphi}.$$

It is equal to the function f_1 in the proof of Theorem 2.1. By Theorem 3.1, the stationary point O of the system (4.1) is monodromic for $|3a - 2c| < 2\sqrt{6}$. The truncated system (*i.e.*, $b = 0$) is reversible $(x, y, t \to x, -y, -t)$, and its monomials are of degree 12; the degree of the monomial bx^5 equals 13. Hence the system (4.1) has a weak focus by Theorem 3.2, and we have $r_1(2\pi) = 1$ without computing the integral of the function F over the period.

We put $c = 3a/2$ and investigate the limit cycle generation in the system (4.1) under the change of coefficient (parameter) $a > 0$.

The function g_2 (see the formula (2.5)) takes the form

$$g_2(\varphi) = \frac{b\,\cos^7\varphi(2 + \sin^2\varphi)\,(3\,a\,\sin^2\varphi - 2\cos^3\varphi)}{(2\,\cos^6\varphi + 3\,\sin^4\varphi)^2},$$

where $r_1(\varphi)$ is found by the formula (2.4).

The second focal value is found by the formula (2.6) taking the integral over the period $[0, 2\pi]$. We take $b = 1$, since the second focal value $r_2(a, b) = r_2(a)$ depends on b linearly.

Computations show that the system (4.1) has the second order focus for all a except $a = a_0 \approx 4.54132378$, when the focal value $r_2(a)$ changes sign, *i.e.*, $r_2(a) < 0$ for $a < a_0$, and $r_2(a) > 0$ for $a_0 < a$. Besides, $r_2(a) \asymp (a - a_0)C_2$, $r_3(a) \asymp (a - a_0)^2 C_3$ and $r_4(a) \asymp C_4$ as $a \to a_0$, where $r_3(a)$ and $r_4(a)$ are the third and the fourth focal values respectively, and C_2, C_3, C_4 are positive constants, which may be found with the same accuracy as the focal values if we compute the mixed variations of equations (2.3), (2.5) with respect to the parameter a [8].

Consequently, the focus is unstable in the coordinates (3.2) for $a > a_0$, since $r(2\pi) > r(0)$ at the origin; and for $a < a_0$, the focus is stable, since $r(2\pi) < r(0)$ for small $r(0)$. Hence for $a < a_0$, the system (4.1) has limit cycles, which are unstable in coordinates (3.2) (see Fig. 1). In the original coordinates, the stability and instability here interchange, as was remarked earlier.

Let us estimate the *initial radius* of the limit cycle, *i.e.*, the value $r(2\pi) = r(0) = r(0)(a)$ for small $a - a_0 < 0$. We use the expansion (2.7) and obtain the formula

$$r(0)(a) \approx 2\frac{-r_3(a) + \sqrt{r_3^2(a) - 3r_2(a)r_4(a)}}{r_4(a)}, \tag{4.2}$$

i.e., $r(0)(a) \asymp \sqrt{a_0 - a}$.

For example, for $a = a_1 = a_0 - 0.1$ the initial radius of the limit cycle equals $r(0)(a_1) \approx 0.57026001$. Computed focal values are $r_2(a_1) \approx -0.03689080$, $r_3(a_1) \approx 0.00204139$, $r_4(a_1) \approx 1.22542310$. Using the formula (4.2), we obtain $r(0)(a_1) \approx 0.59772201$.

Now, we make the change of variables (3.2) with $\alpha = (1/2)^{1/6}$, $\beta = (1/3)^{1/4}$, $k = 1/3$, $\ell = 1/2$. Here we consider variational equations on the interval $[-\pi, \pi]$.

The first variational equation takes the form

$$r_1'(\varphi) = -\frac{a\sqrt{6}}{12}\,\mathrm{sign}\,(\sin\varphi)\, r_1(\varphi).$$

We substitute the solution to this equation into the second variational equation, and after some simplifications, we obtain the equation for the critical value of the parameter a when the limit cycle is born:

$$0 = \int_0^\pi \frac{|\cos t|^{2/3}(3\,a\cos t\sin t - \sqrt{6}\cos^2 t)\exp(-a\sqrt{6}\,t/12)}{\sqrt{\sin t}}\,dt. \tag{4.3}$$

We found the solution $a = a_0 \approx 4.54132378$ to the equation (4.3) with no less than 30 decimal places. We put forward a hypothesis: the value a_0 is not algebraic. ■

Algebraic solvability of CF problem means that all center conditions are algebraic. This is, generally, not the case [4]. We recall that almost algebraic solvability [4] of CF problem is the algebraic solvability considered for systems with fixed coefficients of monomials of the truncated system.

Theorem 4.1 *Let the system* (3.1) *satisfy the condition of Theorem 3.1 and the coefficients of the truncated system be numbers. Then all center conditions for the system* (3.1) *are polynomial ones.*

Proof. The solution $r_1(\varphi)$ to the first variational equation (2.3) does not depend on the coefficients of the system (3.1) by the data. The denominators in the functions $g_k(\varphi)$ do not depend on them either (see the proof of Theorem 3.1). Hence if the previous center conditions are satisfied, *i.e.*, the functions $r_j(\varphi)$, $j = 1, \ldots, k-1$, are 2π-periodic, then the function $g_k(\varphi)$ is a combination of 2π-periodic functions with coefficients of polynomials in coefficients of the system (3.1). ■

For polynomial systems (3.1) there may be, obviously, only a finite number of independent center conditions by virtue of Hilbert's theorem on the bases in polynomial ideals [7].

We remark that Theorem 2 in [4, page. 36] is a special case of Theorem 4.1 ($m = n$).

The evidence suggests that an analytical Poincaré mapping may exist only for systems with one edge of the Newton polygon. This may be the reason why systems with more than one edge of the Newton polygon were never studied in the framework of CF problem until recently [1, 2].

However, the changes of variables (3.2) may be applied in curvilinear sectors on Riemann surfaces instead on the plane $\mathbb{R}^2$. One such example was studied in [10] for an ODE system with a Hamiltonian truncation. For the first time, the center conditions were obtained for the system with two edges of the Newton polygon along with the explicit asymptotics of the Poincaré mapping.

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