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On Modulus of Smoothness of Functions Given on Compact Groups

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Abstract

The connection between structural and constructive characterization of a function is one of the most important problems of approximation theory and this relations is given by Jackson type theorems. The structural properties of functions, *i.e.*, the modulus of smoothness can be given by different methods.

In this article we define different moduli of smoothness of order k for functions and study the relations between these moduli of smoothness and give some unsolved problems on equivalence of these definitions. Also we will prove some Jackson type theorems on compact groups.

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1 Introduction

We know that most of the theorems about approximation of 2π-periodic functions have a natural analogue in the approximation of functions on the sphere $\mathbb{S}^n$ by spherical polynomials and these results would be a starting point for the approximation theory on compact groups .

2 Preliminaries and notations

Let G be a compact group with dual space $\hat{G}$, and dg denote the Haar-measure on G normalized by the condition $\int_G dg = 1$, and $\int_G f(g)\, dg$ denote the Haar integral of a function f on G. Let U_α, $\alpha \in \hat{G}$, denote the irreducible unitary representation of G in the finite dimensional Hilbert space V_α. We reserve the symbol d_α for the dimension of U_α . Thus d_α is a positive integer. Also, we denote by χ_α and t_{ij}^α $(i, j = 1, 2, \ldots, d_\alpha)$, $\alpha \in \hat{G}$, the character and matrix elements (coordinate functions) of U_α respectively.

Let $L_p(G)$ be the space of all functions f equipped with the norm

$$\|f\|_p = \left\{ \int_G |f(g)|^p \, dg \right\}^{1/p} .$$

We write $\|.\|_p$ instead of $\|.\|_{L_p(G)}$ and $L_\infty = C$ is the corresponding space of continuous functions and $\|f\| = \max\{|f(g)| : \ g \in G\}$. As it is known (see [1] or [9], p. 99) the space $L_2(G)$ can be decomposed into the sum:

$$L_2(G) = \sum_{\alpha \in \hat{G}} \oplus H_\alpha \ ,$$

where

$$H_\alpha = \{f \in C(G) : \ f(g) = \operatorname{tr}(U_\alpha(g)C), \ C = \operatorname{Hom}(V_\alpha, V_\alpha)\}.$$

This theorem is one of the most important results of harmonic analysis on compact groups.

The orthogonal projection $Y_\alpha : \ L_2(G) \longrightarrow H_\alpha$ is given by the formula

$$(Y_\alpha f)(g) = d_\alpha \int_G f(h) \chi_\alpha(gh^{-1}) \, dh, \tag{1}$$

where $(Y_\alpha f)(g)$ does not depends on the choice of a basis in L_2. Carrying out this construction for every space $H_\alpha, \ \alpha \in \hat{G}$, we obtain an orthonormal basis in L_2 consisting of the functions $\{\sqrt{d_\alpha} t_{ij}^\alpha : \ \alpha \in \hat{G}, \ 1 \leq i, j \leq d_\alpha\}$. Any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this basis:

$$f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g), \tag{2}$$

where the Fourier coefficients a_{ij}^α are defined by the following relations

$$a_{ij}^\alpha = d_\alpha \int_G f(g) \overline{t_{ij}^\alpha(g)} \, dg, \tag{3}$$

so that $\overline{t_{ij}^\alpha(g)} = t_{ij}^\alpha(g^{-1})$, where g^{-1} is the inverse of g.

Note that the series (2) is a convergent series in the mean and the Parseval equality

$$\int_G |f(g)|^2 \, dg = \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2$$

holds. The aforementioned result of harmonic analysis on a compact group can be found, for example, in [1, 2, 4] and [9].

We denote by Sh_u the generalized translation operator on the compact group G defined by

$$(Sh_u f)(g) = \int_G f(tut^{-1}g)\,dt,$$

$$(\triangle_u f)(g) = f(g) - (Sh_u f)(g) = (E - Sh_u)f,$$

where $u, g \in G$ and E is the identity operator.

We set

$$\triangle_u^k f = \triangle_u(\triangle_u^{k-1} f) = (E - Sh_u)^k f = \sum_{i=0}^{k} (-1)^{k+i} C_k^i Sh_u^i f \tag{4}$$

in which $Sh_u^0 f = f,\ Sh_u(Sh_u^{i-1} f) = Sh_u^i f,\ i = 1, 2, \ldots, k,\ k \in \mathbb{N}$. We note that α is a complicated index. Since $\hat{G}$ is a countable set, there are only countably many $\alpha \in \hat{G}$ for which $\alpha_{ij}^{\alpha} \neq 0$ for some i and j; enumerate them as $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$. So $d_{\alpha_0} < d_{\alpha_1} < d_{\alpha_2} < \cdots < d_{\alpha_n} < \cdots$. Because of that, the symbol "$\alpha < n$" is interpreted as $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \subset \hat{G}$, and $\alpha \geq n$ denotes the set $\hat{G} \backslash (\alpha < n)$. Let d_α, as usual, be the dimension of H_α. For typographical convenience we will write d_n for the dimension of the representation $U^{\alpha_n},\ n = 1, 2, \ldots$ (see [7] or [8]).

$E_n(f)_p$ will denote the approximation of the function $f \in L_p(G)$ by "spherical" polynomials of degree not greater than n;

$$E_n(f)_p = \inf\{\|f - T_n\|_p :\ T_n \in \sum_{\alpha < n, \alpha \in \hat{G}} \oplus H_\alpha\}\ .$$

The sequence $\{E_n(f)_p\}_{n=0}^{\infty}$ of best approximations is a constructive characteristic of the function f.

In the capacity of structural characteristic of the function f on a compact group G, we define its spherical modulus of smoothness of order k by

$$\omega_k(f; \tau)_p = \sup\{\|(E - Sh_u)^k f\|_p :\ u \in W_\tau\},$$

where W_τ is a neighbourhood of e in G. In other words,

$$W_\tau = \{u :\ \rho(u, e) < \tau,\ u \in G\},$$

where ρ is a pseudo-metric on G and τ is any positive real number (see [8]).

Now we define the "iterated modulus of smoothness" in the L_p-metric by

$$\omega_k^*(f; \tau)_p = \sup\{\|\triangle_{u_1}(\triangle_{u_2}(\ldots \triangle_{u_k} f(g)))\|_p :\ u_i \in W_{\tau_i},\ 0 \leq \tau_i < \tau\},$$

where $W_{\tau_i} = \{u_i :\ \rho(u_i, e) < \tau_i,\ u_i \in G\}$.

3 Main results

The following theorem seems to be the most important case of the approximation theory on compact groups.

Theorem 1 *The following equality holds for all $u_i,\, g \in G,\ 1 \le i \le k$:*

$$(Sh_{u_1}(Sh_{u_2}(\dots Sh_{u_k}f)))(g) = \int_G dt_1 \int_G dt_2 \cdots \int_G f(t_1u_1t_1^{-1}t_2u_2t_2^{-1}\dots t_ku_kt_k^{-1}g)\, dt_k$$

$$= \oint_G f(\Pi_{i=1}^k t_iu_it_i^{-1}g)\Pi_{i=1}^k\, dt_i,$$

where

$$\oint_G = \underbrace{\int_G\int_G\cdots\int_G}_{k}.$$

Proof: First of all, it is clear that

$$(Sh_{u_1}[Sh_{u_2}f])(g) = (Sh_{u_1}[\int_G f(t_2u_2t_2^{-1}g)\, dt_2])(g)$$

$$= \int_G\int_G f(t_2u_2t_2^{-1}t_1u_1t_1^{-1}g)\, dt_2\, dt_1 = \int_G\int_G f(t_1u_1t_1^{-1}t_2u_2t_2^{-1}g)\, dt_1\, dt_2.$$

In the last equality we used the invariance of Haar measure. From this by using the mathematical induction, the proof of the theorem is complete.

Lemma 1 *For the matrix elements, the following equality holds:*

$$\oint_G t_{ij}^{\alpha}(\Pi_{i=1}^k t_iu_it_i^{-1}g)\Pi_{i=1}^k\, dt_i = \frac{\chi_\alpha(u_1)\chi_\alpha(u_2)\dots\chi_\alpha(u_k)t_{ij}^{\alpha}(g)}{d_\alpha^k}.$$

Proof: We have proved the following equality in [8]:

$$\int_G t_{ij}^{\alpha}(tut^{-1}g)dt = \frac{\chi_\alpha(u)}{d_\alpha}t_{ij}^{\alpha}(g),\ u \in G,\ g \in G.$$

Now, by using the mathematical induction the proof of the lemma is complete.

Corollary 1 *If $u_1 = u_2 = \dots = u_k = u$, then*

$$\oint_G t_{ij}^{\alpha}(\Pi_{i=1}^k t_iut_i^{-1}g)\Pi_{i=1}^k\, dt_i = \left[\frac{\chi_\alpha(u)}{d_\alpha}\right]^k t_{ij}^{\alpha}(g).$$

Corollary 2 *The following equality holds for characters of the representation:*

$$\oint_G \chi_\alpha(\Pi_{i=1}^k t_i u_i t_i^{-1} g)\Pi_{i=1}^k \, dt_i = \frac{\chi_\alpha(u_1)\chi_\alpha(u_2)\cdots\chi_\alpha(u_k)}{d_\alpha^k}\chi_\alpha(g).$$

Corollary 3

$$\oint_G \chi_\alpha(\Pi_{i=1}^k t_i u t_i^{-1} g)\Pi_{i=1}^k dt_i = \left[\frac{\chi_\alpha(u)}{d_\alpha}\right]^k \chi_\alpha(g).$$

Putting $k = 1$ in Corollary 3, we obtain the known Weyl formula (see [5]):

$$\int_G \chi_\alpha(tut^{-1}g)\, dt = \frac{\chi_\alpha(u)}{d_\alpha}\chi_\alpha(g).$$

The following are simple facts with frequent usage (see [8]):

If $f \in L_p$, then

1) $\|Sh_u f\|_p \le \|f\|_p$.

2) $\|f - Sh_u f\|_p \longrightarrow 0$ as $u \longrightarrow e$.

3) $(Y_\alpha(Sh_u f))(g) = \frac{\chi_\alpha(u)}{\chi_\alpha(e)}(Y_\alpha f)(g),\ \forall \alpha \in \hat{G}$.

We note that $\chi_\alpha(e) = d_\alpha$.

It is not hard to see that the following analogous properties hold for $f \in L_p$.

1′) $\|(Sh_{u_1}(Sh_{u_2}(\dots Sh_{u_k} f)))\|_p \le \|f\|_p$.

(The proof of this property follows from Theorem 1) .

2′) $\|f - (Sh_{u_1}(Sh_{u_2}(\dots Sh_{u_k} f)))\|_p \longrightarrow 0$ as $\tau_i \longrightarrow 0$.

3′) $(Y_\alpha(Sh_{u_1}(Sh_{u_2}(\dots Sh_{u_k} f))))(g) = \frac{\chi_\alpha(u_1)\chi_\alpha(u_2)\dots\chi_\alpha(u_k)}{d_\alpha^k}(Y_\alpha f)(g), \forall \alpha \in \hat{G}$.

Theorem 2 *If $f \in L_2$ and f is not constant, then*

$$E_n(f)_2 \le \sqrt{\frac{d_n}{d_n - 2k}}\omega_k^*(f; \frac{1}{n})_2\,, \quad n = 1, 2, \dots$$

Proof: Let $f \in L_2$ and $S_n(f, g)$ denote the n-th partial sum of Fourier series (2), *i.e.*,

$$S_n(f, g) = \sum_{\alpha<n} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g) = \sum_{p=0}^{n} \sum_{i,j=1}^{d_{\alpha_p}} a_{ij}^{\alpha_p} t_{ij}^{\alpha_p}(g).$$

Using Parseval's equality for the compact group G, we have

$$E_n^2(f)_2 = \|f - S_n(f)\|_2^2 = \sum_{\alpha\ge n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2.$$

Using 3′), it is not hard to see that

$$\begin{aligned}
&(Y_\alpha(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f))))(g)\\
= &\left(1-\frac{\chi_\alpha(u_1)}{d_\alpha}\right)\left(1-\frac{\chi_\alpha(u_2)}{d_\alpha}\right)\cdots\left(1-\frac{\chi_\alpha(u_k)}{d_\alpha}\right)(Y_\alpha f)(g), \quad \alpha\in\hat{G}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f)))(g)\\
= &\sum_{\alpha\in\hat{G}}\left(1-\frac{\chi_\alpha(u_1)}{d_\alpha}\right)\left(1-\frac{\chi_\alpha(u_2)}{d_\alpha}\right)\cdots\left(1-\frac{\chi_\alpha(u_k)}{d_\alpha}\right)\sum_{i,j=1}^{d_\alpha}a_{ij}^\alpha t_{ij}^\alpha(g).
\end{aligned}$$

By another application of Parseval's equality we obtain

$$\begin{aligned}
&\|(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f)))(g)\|_2^2\\
= &\sum_{\alpha\in\hat{G}}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(|1-\frac{\chi_\alpha(u_1)}{d_\alpha}||1-\frac{\chi_\alpha(u_2)}{d_\alpha}|\cdots|1-\frac{\chi_\alpha(u_k)}{d_\alpha}|\right)^2|a_{ij}^\alpha|^2\\
\geq &\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(|1-\frac{\chi_\alpha(u_1)}{d_\alpha}||1-\frac{\chi_\alpha(u_2)}{d_\alpha}|\ldots|1-\frac{\chi_\alpha(u_k)}{d_\alpha}|\right)^2|a_{ij}^\alpha|^2\\
= &\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(1-\frac{2\mathrm{Re}\,\chi_\alpha(u_1)}{d_\alpha}+\frac{|\chi_\alpha(u_1)|^2}{{d_\alpha}^2}\right)\left(1-\frac{2\mathrm{Re}\,\chi_\alpha(u_2)}{d_\alpha}+\frac{|\chi_\alpha(u_2)|^2}{{d_\alpha}^2}\right)\\
&\cdots\left(1-\frac{2\mathrm{Re}\,\chi_\alpha(u_k)}{d_\alpha}+\frac{|\chi_\alpha(u_k)|^2}{{d_\alpha}^2}\right)|a_{ij}^\alpha|^2.
\end{aligned}$$

Now using Bernoulli's inequality

$$(1+x_1)(1+x_2)\cdots(1+x_k)\geq 1+x_1+x_2+\cdots+x_k$$

for $x_i\geq -1,\ i=1,2,\ldots,k$, and putting

$$x_i=\frac{|\chi_\alpha(u_i)|^2}{{d_\alpha}^2}-\frac{2\mathrm{Re}\,\chi_\alpha(u_i)}{d_\alpha}, \quad i=1,2,\ldots,k,$$

we obtain

$$\begin{aligned}
&\|(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f)))(g)\|_2^2\\
\geq &\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(1+\sum_{l=1}^{k}\left(\frac{|\chi_\alpha(u_l)|^2}{{d_\alpha}^2}-\frac{2\mathrm{Re}\,\chi_\alpha(u_l)}{d_\alpha}\right)\right)|a_{ij}^\alpha|^2\\
\geq &\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(1-2\sum_{l=1}^{k}\frac{\mathrm{Re}\,\chi_\alpha(u_l)}{d_\alpha}\right)|a_{ij}^\alpha|^2\,.
\end{aligned}$$

Consequently,

$$\begin{aligned} &\|(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f)))(g)\|_2^2 \\ \geq\ & \sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}|a_{ij}^\alpha|^2-\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\sum_{l=1}^{k}\frac{2\mathrm{Re}\,\chi_\alpha(u_l)}{d_\alpha}|a_{ij}^\alpha|^2\,, \end{aligned}$$

therefore,

$$\begin{aligned} E_n^2(f)_2 &\leq \|(\triangle_{u_1}(\triangle_{u_2}(\ldots\triangle_{u_k}f)))(g)\|_2^2 \\ &+ 2\sum_{\alpha\geq n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\sum_{l=1}^{k}\frac{2\mathrm{Re}\,\chi_\alpha(u_l)}{d_\alpha}|a_{ij}^\alpha|^2\,. \end{aligned} \tag{5}$$

By repeating the method of [8], it is not hard to see that

$$E_n^2(f)_2\leq(\omega_k^*(f;\frac{1}{n})_2)^2+\frac{2k}{d_n}E_n^2(f)_2.$$

Finally, we obtain

$$E_n(f)_2\leq\sqrt{\frac{d_n}{d_n-2k}}\omega_k^*(f;\frac{1}{n})_2,$$

which proves the theorem.

This theorem is given in [6] for the case $k=1$.

Remark: The problem of modulus of smoothness of functions given on the sphere has been studied by P. I. Lizorkin and S. M. Nikol'skii, Kh. P. Rustamov, M. Wehrens [10] and G. A. Kalyabin [3].

We note that the question of the equivalence of modulus of smoothness (for $k>1$) remains open for $1\leq p\leq\infty$.

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