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On Poisson Semigroup Generated by the Generalized B-Translation

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Abstract

The Poisson semigroup associated with the singular differential operator $\Delta_B = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{2\upsilon_k}{x_k} \cdot \frac{\partial}{\partial x_k} \right)$ is introduced and some properties are studied.

1 Auxiliary definitions, notations and results

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x \cdot \xi) = x_1\xi_1 + \cdots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$ and $\mathbb{R}_+^n = \{x = (x_1, \ldots, x_n)\, ;\ x_1 > 0, \ldots, x_n > 0\}$.

We denote by $\Delta_B \equiv \Delta_B(x)$, $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$ the singular differential operator

$$\Delta_B = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{2\upsilon_k}{x_k} \cdot \frac{\partial}{\partial x_k} \right) \quad (\upsilon_1 > 0, \ldots, \upsilon_n > 0)\,. \tag{1.1}$$

Let $L_{p,\upsilon} \equiv L_p\left(\mathbb{R}_+^n, x^{2\upsilon}\, dx\right)$, $1 \le p < \infty$, be the space of measurable functions on $\mathbb{R}_+^n$ with the norm

$$\|f\|_{p,\upsilon} = \left(\int\limits_{\mathbb{R}_+^n} |f(x)|^p\, x^{2\upsilon}\, dx \right)^{1/p} ; \quad x^{2\upsilon} = x_1^{2\upsilon_1} \ldots x_n^{2\upsilon_n};\ dx = dx_1 \ldots dx_n. \tag{1.2}$$

For $x \in \mathbb{R}_+^n$, $y \in \mathbb{R}_+^n$, the generalized B-translation of $f :\ \mathbb{R}_+^n \longrightarrow C$ is defined by

$$T^y f(x) = \pi^{-n/2} \prod_{k=1}^{n} \Gamma\left(\upsilon_k + \frac{1}{2}\right) \Gamma^{-1}(\upsilon_k) \int\limits_0^{\infty} \cdots \int\limits_0^{\infty} \prod_{k=1}^{n} \sin^{2\upsilon_k - 1} \alpha_k \tag{1.3}$$
$$\times \quad f\left(\sqrt{x_1^2 - 2x_1y_1\cos\alpha_1 + y_1^2}, \ldots, \sqrt{x_n^2 - 2x_ny_n\cos\alpha_n + y_n^2} \right) d\alpha_1 \ldots d\alpha_n.$$

For the relevant one-dimensional generalized Bessel translation operator

$$S^{\rho}g(r)=\frac{\Gamma\left(\upsilon+\frac{1}{2}\right)}{\Gamma\left(\upsilon\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_{0}^{\pi}g\left(\sqrt{r^{2}-2r\rho\cos\alpha+\rho^{2}}\right)\sin^{2\upsilon-1}\alpha\,d\alpha$$

the following relations are known [3]:

$$S^{\rho}g\left(r\right)=S^{r}g\left(\rho\right),\quad S^{\rho}S^{\tau}g\left(r\right)=S^{\tau}S^{\rho}g\left(r\right),$$
$$S^{\rho}g\left(r\right)=S^{-\rho}g\left(r\right),\qquad S^{0}g\left(r\right)=g\left(r\right),$$

$$\int\limits_{0}^{\infty}f\left(r\right)S^{r}g\left(\rho\right)r^{2\upsilon}\,dr=\int\limits_{0}^{\infty}S^{r}f\left(\rho\right)g\left(r\right)r^{2\upsilon}\,dr.$$

Let $f\in L_{p,\upsilon}$, $1\leq p<\infty$. Then for all $x\in\mathbb{R}_{+}^{n}$, the function $T^{x}f$ belongs to $L_{p,\upsilon}$ (see [4]) and

$$\left\|T^{x}f\right\|_{p,\upsilon}\leq\left\|f\right\|_{p,\upsilon}. \tag{1.4}$$

The generalized B-translation operator T^{y} generates the corresponding B-convolution

$$\left(f\otimes g\right)\left(x\right)=\int\limits_{\mathbb{R}_{+}^{n},}T^{y}f\left(x\right)g\left(y\right)y^{2\upsilon}\,dy. \tag{1.5}$$

By using (1.4) and the Riesz-Thorin interpolation theorem it is not difficult to prove the corresponding Young inequality

$$\left\|f\otimes g\right\|_{r,\upsilon}\leq\left\|f\right\|_{p,\upsilon}.\left\|g\right\|_{q,\upsilon},\quad 1\leq p,q,r\leq\infty,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1. \tag{1.6}$$

The Fourier-Bessel transform and its inverse are defined by

$$\left(F_{\upsilon}\varphi\right)\left(z\right)=\int\limits_{\mathbb{R}_{+}^{n}}\varphi\left(x\right)\left(\prod_{k=1}^{n}j_{\upsilon_{k}-\frac{1}{2}}\left(x_{k}z_{k}\right)\right)x^{2\upsilon}\,dx, \tag{1.7}$$

$$\left(F_{\upsilon}^{-1}\varphi\right)\left(x\right)=c_{\upsilon}\left(n\right)\left(F_{\upsilon}\varphi\right)\left(-x\right),\quad c_{\upsilon}\left(n\right)=\left(\prod_{k=1}^{n}2^{2\upsilon_{k}}\Gamma^{2}\left(\upsilon_{k}+\frac{1}{2}\right)\right)^{-1}, \tag{1.8}$$

where $j_{p}\left(t\right)$ $\left(t>0,\ p>-\frac{1}{2}\right)$ is connected with the Bessel function of the first kind $J_{p}\left(t\right)$ as follows [3]:

$$j_{p}\left(t\right)=2^{p}\Gamma\left(p+1\right)\frac{J_{p}\left(t\right)}{t^{p}}.$$

It is known that for $f\in L_{p,\upsilon}$ $(p=1$ or $p=2)$

$$F_{\upsilon}\left(f\otimes g\right)=F_{\upsilon}\left(f\right)F_{\upsilon}\left(g\right). \tag{1.9}$$

2 A Poisson semigroup associated with the generalized B-translation and its properties

The Poisson semigroup associated with Δ_B is an integral operator of convolution type generated by the generalized B-translation. The kernel of this operator is defined as the Fourier-Bessel transform of the function $\exp(-\alpha|y|)$ $\left(y \in \mathbb{R}_+^n,\ \alpha > 0\right)$.

Let us calculate $F_\upsilon(\exp(-\alpha|y|))$ by using the following formulas [5] and [1], respectively,

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}}{\sqrt{t}} e^{-\beta^2/4t}\, dt,$$

$$F_\upsilon\left(e^{-\alpha|x|^2}\right)(t) = 2^{-n}\Gamma\left(\upsilon_1 + \frac{1}{2}\right)\dots\Gamma\left(\upsilon_n + \frac{1}{2}\right)\alpha^{-\frac{2\upsilon_1+\dots+2\upsilon_n+n}{2}} e^{-\frac{|t|^2}{4\alpha}}.$$

By Fubini's theorem we have

$$F_\upsilon\left(e^{-|y|}\right)(x) = \int_{\mathbb{R}_+^n} e^{-|y|}\left(\prod_{k=1}^n j_{\upsilon_k-\frac{1}{2}}(x_k y_k)\, y_k^{2\upsilon_k}\right) dy$$

$$= \int_{\mathbb{R}_+^n}\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}} e^{-|y|^2/4t}\, dt\right)\left(\prod_{k=1}^n j_{\upsilon_k-\frac{1}{2}}(x_k y_k)\, y_k^{2\upsilon_k}\right) dy$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}} \int_{\mathbb{R}_+^n} e^{-|y|^2/4t} \prod_{k=1}^n j_{\upsilon_k-\frac{1}{2}}(x_k y_k)\, y_k^{2\upsilon_k}\, dy\, dt$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}} \prod_{k=1}^n \int_0^\infty e^{-y_k^2/4t} j_{\upsilon_k-\frac{1}{2}}(x_k y_k)\, y_k^{2\upsilon_k}\, dy_k\, dt$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}} \prod_{k=1}^n \frac{1}{2}\left(\frac{1}{4t}\right)^{-\upsilon_k-\frac{1}{2}}\Gamma\left(\upsilon_k+\frac{1}{2}\right) e^{-(x_k^2/4)\frac{1}{4t}}\, dt$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\left[\frac{1}{2^n}(4t)^{\upsilon_1+\frac{1}{2}}\dots(4t)^{\upsilon_n+\frac{1}{2}}\Gamma(\upsilon_1+\frac{1}{2})\dots\Gamma(\upsilon_n+\frac{1}{2})e^{-tx_1^2}\dots e^{-tx_n^2}\right] dt$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{2^n}\cdot 2^{2\upsilon_1+\dots+2\upsilon_n}2^n\Gamma\left(\upsilon_1+\frac{1}{2}\right)\dots\Gamma\left(\upsilon_n+\frac{1}{2}\right)\int_0^\infty \frac{e^{-t}}{\sqrt{t}} t^{\upsilon_1+\dots+\upsilon_n+\frac{n}{2}} e^{-t|x|^2}\, dt$$

$$= \frac{1}{\sqrt{\pi}}\, 2^{2\upsilon_1+\cdots+2\upsilon_n} \Gamma\left(\upsilon_1 + \frac{1}{2}\right) \ldots \Gamma\left(\upsilon_n + \frac{1}{2}\right) \int\limits_0^\infty t^{\frac{n-1}{2}+\upsilon_1+\cdots+\upsilon_n} e^{-t\left(1+|x|^2\right)}\, dt$$

$$= \frac{1}{\sqrt{\pi}}\, 2^{\upsilon_1+\cdots+\upsilon_n} \left(\sqrt{c_\upsilon(n)}\right)^{-1} \int\limits_0^\infty t^{\frac{n+1}{2}+\upsilon_1+\cdots+\upsilon_n-1} e^{-t\left(1+|x|^2\right)}\, dt,$$

where $c_\upsilon(n)$ is defined in (1.8).

From the last equality we get

$$F_\upsilon\left(e^{-|y|}\right)(x) = \frac{1}{\sqrt{\pi}} 2^{\upsilon_1+\cdots+\upsilon_n} \left(\sqrt{c_\upsilon(n)}\right)^{-1} \Gamma\left(\frac{n+1}{2} + \upsilon_1 + \cdots + \upsilon_n\right) \times \left(1+|x|^2\right)^{-\left(\frac{n+1}{2}+\upsilon_1+\cdots+\upsilon_n\right)},$$

by using the definition of the Gamma function.

Finally, using the equality

$$F_\upsilon(f(\lambda y))(x) = \lambda^{-(n+2\upsilon_1+\cdots+2\upsilon_n)} F_\upsilon(f(y))\left(\frac{x}{\lambda}\right) \quad (\lambda > 0),$$

we have

$$F_\upsilon\left(e^{-\alpha|y|}\right)(x) = \frac{1}{\sqrt{\pi}} 2^{\upsilon_1+\cdots+\upsilon_n} \left(\sqrt{c_\upsilon(n)}\right)^{-1} \Gamma\left(\frac{n+1}{2} + \upsilon_1 + \cdots + \upsilon_n\right) \times \frac{\alpha}{\left(|x|^2+\alpha^2\right)^{\frac{n+1}{2}+\upsilon_1+\cdots+\upsilon_n}}.$$

In view of the last equality we define the Poisson kernel as

$$P_\upsilon(x;\alpha) = \sqrt{c_\upsilon(n)} \frac{1}{\sqrt{\pi}} 2^{\upsilon_1+\cdots+\upsilon_n} \Gamma\left(\frac{n+1}{2} + \upsilon_1 + \cdots + \upsilon_n\right) \times \frac{\alpha}{\left(|x|^2+\alpha^2\right)^{\frac{n+1}{2}+\upsilon_1+\cdots+\upsilon_n}}. \tag{2.1}$$

It is not hard to verify the following properties of $P_\upsilon(x;\alpha)$:

1) $F_\upsilon(P_\upsilon(x;\alpha))(x) = e^{-\alpha|x|}$;

2) $\|P_\upsilon(\cdot;\alpha)\|_{1,\upsilon} = 1 \quad$ (for all $\alpha > 0$);

3) $P_\upsilon(x;\alpha+\beta) = P_\upsilon(x;\alpha) \otimes P_\upsilon(x;\beta) \equiv \int\limits_{\mathbb{R}_+^n} P_\upsilon(y;\alpha)\, T_x^y(P_\upsilon(x;\beta))\, y^{2\upsilon}\, dy,$

where the symbol T_x^y denotes the translation T^y applied to the variable x.

Now, we define the Poisson integral (semigroup) generated by the generalized translation as

$$(V_\alpha f)(x) \equiv v(x;\alpha) = \int\limits_{\mathbb{R}_+^n} f(y)\, T_x^y \left(P_\upsilon(x;\alpha)\right) y^{2\upsilon}\, dy. \tag{2.2}$$

By making use of the Fourier-Bessel transform, it is not difficult to verify that the Poisson integral $v(x;\alpha)$ is the solution of the following boundary value problem

$$\begin{cases} \left(\frac{\partial^2}{\partial \alpha^2} + \Delta_B(x)\right) v(x;\alpha) = 0, \\ v(x;\alpha)\,|_{\alpha=0} = f(x) \end{cases}$$

for “good” f.

It is easy to show the semigroup property, $V_\alpha V_\beta = V_{\alpha+\beta}$ $(0 < \alpha,\ \beta < \infty)$ of $\{V_\alpha\}$, $\alpha > 0$ by using the Fourier-Bessel transform

$$F_\upsilon(V_{\alpha+\beta} f) = e^{-(\alpha+\beta)|y|} F_\upsilon f = e^{-\alpha|y|}\left(e^{-\alpha|y|} F_\upsilon f\right) = F_\upsilon(V_\alpha V_\beta f).$$

Now, let us investigated the approximation and other properties of the Poisson semigroup $V_\alpha f$ $(\alpha > 0)$ associated with Δ_B. For this, first we define the Hardy-Littlewood maximal function $M_B f$ generated by the generalized translation. The maximal function $M_B f$ of $f \in L_{p,\upsilon}$, $1 \le p \le \infty$, is defined as

$$M_B f\ (x) = \sup_{r>0} \frac{1}{|E_+(0,r)|} \int\limits_{E_+(0,r)} T^y |f(x)|\, y^{2\upsilon}\, dy,$$

where $E_+(x,r) = \left\{y:\ y \in \mathbb{R}_+^n,\ |x-y| < r\right\}$,

$$|E_+(0,r)| = \int\limits_{E_+(0,r)} y^{2\upsilon}\, dy = w(n,\upsilon)\, r^{n+2\upsilon_1+\cdots+2\upsilon_n},\ w(n,\upsilon) = \int\limits_{E_+(0,1)} x^{2\upsilon}\, dx.$$

It is known (see [2]) that the maximal operator M_B is of weak type (1,1) and is bounded on $L_{p,\upsilon}$, $1 < p \le \infty$.

We can prove the following lemma for the maximal function $M_B f$ by using some ideas in E. Stein and G. Weiss's monograph [5].

Lemma 1 *Let* $\varphi \in L_{1,\upsilon}$ *be a radial and* $\psi(r) = \varphi(x)\,|_{|x|=r}$ $(0 < r < \infty)$ *be a non-negative and decreasing function on* $[0,\infty)$*. Then for every* $f \in L_{p,\upsilon}$ $(1 \le p \le \infty)$ *we have*

$$\sup_{\varepsilon>0} |(f \otimes \varphi_\varepsilon)(x)| \le \|\varphi\|_{1,\upsilon}\, M_B f(x), \tag{2.3}$$

where $\varphi_\varepsilon(x) = \varepsilon^{-n-(2\upsilon_1+\cdots+2\upsilon_n)} \varphi\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$.

Proof. For the sake of natural simplicity, we assume $f \geq 0$.
Step I. Let the function φ be defined by

$$\varphi(x) = \begin{cases} \frac{1}{w(\upsilon,n)} & , \quad x \in E_+(0,1), \\ 0 & , \quad x \in \mathbb{R}^n_+ \backslash E_+(0,1). \end{cases}$$

We have $\|\varphi\|_{1,\upsilon} = 1$. Putting $\varphi_\varepsilon(x) = \varepsilon^{-n-(2\upsilon_1+\cdots+2\upsilon_n)}\varphi\left(\frac{x}{\varepsilon}\right)$, we get $\|\varphi_\varepsilon\|_{1,\upsilon} = \|\varphi\|_{1,\upsilon} = 1$ for all $\varepsilon > 0$. Then

$$M_B f(x) = \sup_{\varepsilon>0} |(f \otimes \varphi_\varepsilon)(x)| \quad \text{for all} \quad f \in L_{p,\upsilon} \quad (f \geq 0).$$

Step II. Let $\varphi(x) = \sum\limits_{k=1}^{m} c_k \chi_k(x) \quad (c_k \geq 0, \ k = 1, \ldots, m)$, where $\chi_k(x)$ is the characteristic function of the sphere $E_+(0, r_k)$. Putting $\varphi_\varepsilon(x) = \varepsilon^{-n-(2\upsilon_1+\cdots+2\upsilon_n)}\varphi(\frac{x}{\varepsilon})$, we get

$$(f \otimes \varphi_\varepsilon)(x) = \sum_{k=1}^{m} c_k \varepsilon^{-n-(2\upsilon_1+\cdots+2\upsilon_n)} \int\limits_{E_+(0,\varepsilon r_k)} T^y f(x)\, y^{2\upsilon}\, dy$$

$$\begin{aligned}
&= \sum_{k=1}^{m} c_k w(n,\upsilon)\, r_k^{n+(2\upsilon_1+\cdots+2\upsilon_n)} \frac{1}{w(n,\upsilon)(\varepsilon r_k)^{n+(2\upsilon_1+\cdots+2\upsilon_n)}} \int\limits_{E_+(0,\varepsilon r_k)} T^y f(x)\, y^{2\upsilon}\, dy \\
&\leq M_B f(x) \sum_{k=1}^{m} c_k w(n,\upsilon)\, r_k^{n+(2\upsilon_1+\cdots+2\upsilon_n)} \\
&= M_B f(x) \sum_{k=1}^{m} c_k \int\limits_{E_+(0,r_k)} x^{2\upsilon}\, dx \\
&= M_B f(x) \sum_{k=1}^{m} c_k \int\limits_{\mathbb{R}^n_+} \chi_k(x)\, x^{2\upsilon}\, dx \\
&= M_B f(x) \int\limits_{\mathbb{R}^n_+} \left(\sum_{k=1}^{m} c_k \chi_k(x) \right) x^{2\upsilon}\, dx \\
&= M_B f(x)\, \|\varphi\|_{1,\upsilon}.
\end{aligned}$$

Thus,

$$\sup_{\varepsilon>0} |(f \otimes \varphi_\varepsilon)(x)| \leq \|\varphi\|_{1,\upsilon}\, M_B f(x)$$

for every nonnegative simple function φ.

Step III. Since $\psi(r)$ is nonnegative decreasing on $[0,\infty)$ and the function $\varphi \in L_{1,\upsilon}$ is of the form $\varphi(x) = \psi(|x|)$, then we have $\psi(r) \to 0$ as $r \to \infty$. Thus, it is possible to approximate the nonnegative function $\varphi(x) = \psi(|x|)$ from below by an increasing sequence of simple functions of the type $\varphi_m(x) = \sum_{k=1}^{m} c_k^m \chi_k(x)$. We have proved above the inequality (2.3) for the simple functions φ_m . Now, taking the limit as $m \to \infty$, one concludes the proof. ■

The following theorem states the main result of this work which gives some properties of the Poisson integral generated by the Laplace-Bessel differential operator Δ_B.

Theorem 2 *Let $V_\alpha f$ $(\alpha > 0)$ be the Poisson semigroup for a function f. If $f \in L_{p,\upsilon}$, $1 \le p \le \infty$, then:*

a) $\|V_\alpha f\|_{p,\upsilon} \le \|f\|_{p,\upsilon}$;

b) $\sup_{\alpha>0} |(V_\alpha f)(x)| \le M_B f(x)$;

c) $V_\alpha V_\beta f = V_{\alpha+\beta} f \quad (\alpha > 0,\ \beta > 0)$;

d) $\operatorname{ess}\sup_{x\in\mathbb{R}_+^n} |(V_\alpha f)(x)| \le C\ \alpha^{-\frac{n+2\upsilon_1+\cdots+2\upsilon_n}{p}} \|f\|_{p,\upsilon}$;

e) $(L_{p,\upsilon}) \lim_{\alpha\to 0^+} V_\beta f = f \quad (1 \le p < \infty)$,

where $(L_{p,\upsilon})\lim$ denotes the limit in the norm $L_{p,\upsilon}$ and pointwise for almost all $x \in \mathbb{R}^n$.

Proof.
a) By using Young's inequality (1.6) and the equality $\|P_\upsilon(\cdot;\alpha)\|_{1,\upsilon} = 1$ for all $\alpha > 0$, we have $\|V_\alpha f\|_{p,\upsilon} = \|f \otimes P_\upsilon(\cdot;\alpha)\|_{p,\upsilon} \le \|f\|_{p,\upsilon} \|P_\upsilon(\cdot;\alpha)\|_{1,\upsilon}$.

b) The proof of this result follows directly from (2.3) by taking $\varphi_\varepsilon(x) = P_\upsilon(x;\varepsilon) \equiv \varepsilon^{-n-(2\upsilon_1+\cdots+2\upsilon_n)} P_\upsilon\left(\frac{x}{\varepsilon};1\right)$.

c) is proved above.

d) is obtained by substituting ∞ for r and $P_\upsilon(x;\alpha)$ for the function g in Young's inequality (1.6) and using the homogeneity property of $P_\upsilon(\cdot;\alpha)$.

e) By using the equality $\|P_\upsilon(.;\alpha)\|_{1,\upsilon}=1$ we have

$$\|V_\alpha f - f\|_{p,\upsilon} = \Big\| \int_{\mathbb{R}_+^n} P_\upsilon(x;\alpha)\left[T^x f(y) - f(y)\right] x^{2\upsilon}\, dx \Big\|_{p,\upsilon}.$$

By setting αx instead of x and applying the generalized Minkowski's inequality, we have

$$\|V_\alpha f - f\|_{p,\upsilon} = \Big\| \int_{\mathbb{R}_+^n} P_\upsilon(x;1)\left[T^{\alpha x} f(y) - f(y)\right] x^{2\upsilon}\, dx \Big\|_{p,\upsilon}$$

$$\leq \int_{\mathbb{R}_+^n} P_\upsilon(x;1)\left\|\left[T^{\alpha x} f(y) - f(y)\right]\right\|_{p,\upsilon} x^{2\upsilon}\, dx$$

by using (1.4) and the property

$$\lim_{t\to 0}\left\|T^t f(.) - f(.)\right\|_{p,\upsilon} = 0 \qquad \text{(see [4])}$$

of T^t and, finally, applying the Lebesgue dominated convergence theorem we get

$$\lim_{\alpha\to 0}\|V_\alpha f - f\|_{p,\upsilon} = 0. \qquad \blacksquare$$

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